The given functions are linearly dependent on any interval since
\[ k_1 \sin t + k_2 \cos(t - \pi/2) = 0 \]
for all \( t \) if we choose \( k_1 = 1 \) and \( k_2 = -1 \).

**Example 2**

Show that the functions \( e^t \) and \( e^{2t} \) are linearly independent on any interval.

To establish this result we suppose that
\[ k_1 e^t + k_2 e^{2t} = 0 \quad (3) \]
for all \( t \) in the interval; we must then show that \( k_1 = k_2 = 0 \). Choose two points \( t_0 \) and \( t_1 \) in the interval, where \( t_1 \neq t_0 \). Evaluating Eq. (3) at these points, we obtain
\[ k_1 e^{t_0} + k_2 e^{2t_0} = 0, \]
\[ k_1 e^{t_1} + k_2 e^{2t_1} = 0. \]

The determinant of coefficients is
\[ e^{t_0} e^{2t_1} - e^{2t_0} e^{t_1} = e^{0} e^{t_1} (e^{t_1} - e^{t_0}). \]

Since this determinant is not zero, it follows that the only solution of Eq. (4) is \( k_1 = k_2 = 0 \). Hence \( e^t \) and \( e^{2t} \) are linearly independent.

The following theorem relates linear independence and dependence to the Wronskian.

**Theorem 3.3.1**

If \( f \) and \( g \) are differentiable functions on an open interval \( I \) and if \( W(f, g)(t_0) \neq 0 \) for some point \( t_0 \) in \( I \), then \( f \) and \( g \) are linearly independent on \( I \). Moreover, if \( f \) and \( g \) are linearly dependent on \( I \), then \( W(f, g)(t) = 0 \) for every \( t \) in \( I \).

To prove the first statement in Theorem 3.3.1, consider a linear combination \( k_1 f(t) + k_2 g(t) \), and suppose that this expression is zero throughout the interval. Evaluating the expression and its derivative at \( t_0 \), we have
\[ k_1 f(t_0) + k_2 g(t_0) = 0, \]
\[ k_1 f'(t_0) + k_2 g'(t_0) = 0. \]

The determinant of coefficients of Eqs. (5) is precisely \( W(f, g)(t_0) \), which is not zero by hypothesis. Therefore, the only solution of Eqs. (5) is \( k_1 = k_2 = 0 \), so \( f \) and \( g \) are linearly independent.

The second part of Theorem 3.3.1 follows immediately from the first. Let \( f \) and \( g \) be linearly dependent, and suppose that the conclusion is false, that is, \( W(f, g) \) is not everywhere zero in \( I \). Then there is a point \( t_0 \) such that \( W(f, g)(t_0) \neq 0 \); by the first part of Theorem 3.3.1 this implies that \( f \) and \( g \) are linearly independent, which is a contradiction, thus completing the proof.