Let's take a breather. The previous chapters have seen some heavy going, with sums involving floor, ceiling, mod, phi, and mu functions. Now we're going to study binomial coefficients, which turn out to be (a) more important in applications, and (b) easier to manipulate, than all those other quantities.

5.1 Basic Identities

The symbol \( \binom{n}{k} \) is a binomial coefficient, so called because of an important property we look at later this section, the binomial theorem. But we read the symbol “n choose k.” This incantation arises from its combinatorial interpretation—it is the number of ways to choose a k-element subset from an n-element set. For example, from the set \{1, 2, 3, 4\} we can choose two elements in six ways,

\[
\{1, 2\}, \; \{1, 3\}, \; \{1, 4\}, \; \{2, 3\}, \; \{2, 4\}, \; \{3, 4\};
\]

so \( \binom{4}{2} = 6 \).

To express the number \( \binom{n}{k} \) in more familiar terms it's easiest to first determine the number of k-element sequences, rather than subsets, chosen from an n-element set; for sequences, the order of the elements counts. We use the same argument we used in Chapter 4 to show that n! is the number of permutations of n objects. There are n choices for the first element of the sequence; for each, there are n-1 choices for the second; and so on, until there are \( n-k+1 \) choices for the kth. This gives \( n(n-1) \cdots (n-k+1) \) = \( n^k \) choices in all. And since each k-element subset has exactly k! different orderings, this number of sequences counts each subset exactly k! times. To get our answer, we simply divide by k!:

\[
\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots (1)}.
\]