We can apply this result to the two functions \( f(t) = e^t \) and \( g(t) = e^{2t} \) discussed in Example 2. For any point \( t_0 \) we have

\[
W(f, g)(t_0) = \begin{vmatrix} e^0 & e^{2t_0} \\ e^{2t_0} & 2e^{2t_0} \end{vmatrix} = e^{3t_0} \neq 0.
\]

(6)

The functions \( e^t \) and \( e^{2t} \) are therefore linearly independent on any interval.

You should be careful not to read too much into Theorem 3.3.1. In particular, two functions \( f \) and \( g \) may be linearly independent even though \( W(f, g)(t) = 0 \) for every \( t \) in the interval \( I \). This is illustrated in Problem 28.

Now let us examine further the properties of the Wronskian of two solutions of a second order linear homogeneous differential equation. The following theorem, perhaps surprisingly, gives a simple explicit formula for the Wronskian of any two solutions of any such equation, even if the solutions themselves are not known.

**Theorem 3.3.2** *(Abel’s Theorem)*

If \( y_1 \) and \( y_2 \) are solutions of the differential equation

\[
L[y] = y'' + p(t)y' + q(t)y = 0,
\]

(7)

where \( p \) and \( q \) are continuous on an open interval \( I \), then the Wronskian \( W(y_1, y_2)(t) \) is given by

\[
W(y_1, y_2)(t) = c \exp \left[ -\int p(t) \, dt \right],
\]

(8)

where \( c \) is a certain constant that depends on \( y_1 \) and \( y_2 \), but not on \( t \). Further, \( W(y_1, y_2)(t) \) is either zero for all \( t \) in \( I \) (if \( c = 0 \)) or else is never zero in \( I \) (if \( c \neq 0 \)).

To prove Abel’s theorem we start by noting that \( y_1 \) and \( y_2 \) satisfy

\[
y_1'' + p(t)y_1' + q(t)y_1 = 0,
\]

\[
y_2'' + p(t)y_2' + q(t)y_2 = 0.
\]

(9)

If we multiply the first equation by \( -y_2 \), the second by \( y_1 \), and add the resulting equations, we obtain

\[
(y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1'y_2) = 0.
\]

(10)

Next, we let \( W(t) = W(y_1, y_2)(t) \) and observe that

\[
W' = y_1 y_2'' - y_1'' y_2.
\]

(11)

Then we can write Eq. (10) in the form

\[
W' + p(t)W = 0.
\]

(12)

The result in Theorem 3.3.2 was derived by the Norwegian mathematician Niels Henrik Abel (1802–1829) in 1827 and is known as Abel’s formula. Abel also showed that there is no general formula for solving a quintic, or fifth degree, polynomial equation in terms of explicit algebraic operations on the coefficients, thereby resolving a question that had been open since the sixteenth century. His greatest contributions, however, were in analysis, particularly in the study of elliptic functions. Unfortunately, his work was not widely noticed until after his death. The distinguished French mathematician Legendre called it a “monument more lasting than bronze.”