And now the identities. Our goal in this section will be to learn a few simple rules by which we can solve the vast majority of practical problems involving binomial coefficients.

Definition (5.1) can be recast in terms of factorials in the common case that the upper index $r$ is an integer, $n$, that’s greater than or equal to the lower index $k$:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \text{integers } n \geq k \geq 0. \quad (5.3)$$

To get this formula, we just multiply the numerator and denominator of (5.1) by $(n-k)!$. It’s occasionally useful to expand a binomial coefficient into this factorial form (for example, when proving the hexagon property). And we often want to go the other way, changing factorials into binomials.

The factorial representation hints at a symmetry in Pascal’s triangle: Each row reads the same left-to-right as right-to-left. The identity reflecting this-called the symmetry identity-is obtained by changing $k$ to $n-k$:

$$\binom{n}{k} = \binom{n}{n-k}, \quad \text{integer } n \geq 0, \quad \text{integer } k. \quad (5.4)$$

This formula makes combinatorial sense, because by specifying the $k$ chosen things out of $n$ we’re in effect specifying the $n-k$ unchosen things.

The restriction that $n$ and $k$ be integers in identity (5.4) is obvious, since each lower index must be an integer. But why can’t $n$ be negative? Suppose, for example, that $n = -1$. Is

$$\binom{-1}{k} = \binom{-1}{-1-k}$$

a valid equation? No. For instance, when $k = 0$ we get 1 on the left and 0 on the right. In fact, for any integer $k \geq 0$ the left side is

$$\binom{-1}{k} = \frac{(-1)(-2)\cdots(-k)}{k!} = (-1)^k,$$

which is either 1 or -1; but the right side is 0, because the lower index is negative. And for negative $k$ the left side is 0 but the right side is

$$\binom{-1}{-1-k} = (-1)^{-1-k},$$

which is either 1 or -1. So the equation $\binom{-1}{k} = \binom{-1}{-1-k}$ is always false!

The symmetry identity fails for all other negative integers $n$, too. But unfortunately it’s all too easy to forget this restriction, since the expression in the upper index is sometimes negative only for obscure (but legal) values