property (5.4) with impunity.) Have we been cheating? No. It’s true that
the derivation is valid only for positive integers $r$; but we can claim that the
identity holds for all values of $r$, because both sides of (5.7) are polynomials
in $r$ of degree $k + 1$. A non-zero polynomial of degree $d$ or less can have at
most $d$ distinct zeros; therefore the difference of two such polynomials, which
also has degree $d$ or less, cannot be zero at more than $d$ points unless it is
identically zero. In other words, if two polynomials of degree $d$ or less agree
at more than $d$ points, they must agree everywhere. We have shown that
$(r-k)(;,:) = r(u,;:)$ whenever $r$ is a positive integer; so these two polynomials
agree at infinitely many points, and they must be identically equal.

The proof technique in the previous paragraph, which we will call the
polynomial argument, is useful for extending many identities from integers
to reals; we’ll see it again and again. Some equations, like the symmetry
identity (5.4), are not identities between polynomials, so we can’t always use
this method. But many identities do have the necessary form.

For example, here’s another polynomial identity, perhaps the most im-
portant binomial identity of all, known as the addition formula:

\[
\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}, \quad \text{integer } k. 
\]  

(5.8)

When $r$ is a positive integer, the addition formula tells us that every number
in Pascal’s triangle is the sum of two numbers in the previous row, one directly
above it and the other just to the left. And the formula applies also when $r$
is negative, real, or complex; the only restriction is that $k$ be an integer, so
that the binomial coefficients are defined.

One way to prove the addition formula is to assume that $r$ is a positive
integer and to use the combinatorial interpretation. Recall that $\binom{r}{k}$ is the
number of possible $k$-element subsets chosen from an $r$-element set. If we
have a set of $r$ eggs that includes exactly one bad egg, there are $\binom{r}{k}$ ways to
select $k$ of the eggs. Exactly $\binom{r-1}{k-1}$ of these selections involve nothing but good
eggs; and $\binom{r-1}{k}$ of them contain the bad egg, because such selections have $k-1$
of the $r-1$ good eggs. Adding these two numbers together gives (5.8). This
derivation assumes that $r$ is a positive integer, and that $k \geq 0$. But both sides
of the identity are zero when $k < 0$, and the polynomial argument establishes
(5.8) in all remaining cases.

We can also derive (5.8) by adding together the two absorption identities
(5.7) and (5.6):

\[
(r-k)(;:)+k(;:)=r(;:)+r(;:-1); 
\]

the left side is $r(;:)$, and we can divide through by $r$. This derivation is valid
for everything but $r = 0$, and it’s easy to check that remaining case.