(\binom{5}{3}) = (\binom{4}{3}) + (\binom{4}{2})
= (\binom{3}{3}) + (\binom{3}{2}) + (\binom{4}{2})
= (\binom{2}{3}) + (\binom{2}{2}) + (\binom{3}{2}) + (\binom{4}{2})
= (\binom{1}{3}) + (\binom{1}{2}) + (\binom{2}{2}) + (\binom{3}{2}) + (\binom{4}{2})
= (\binom{0}{3}) + (\binom{0}{2}) + (\binom{1}{2}) + (\binom{2}{2}) + (\binom{3}{2}) + (\binom{4}{2})

Now (\binom{3}{3}) is zero (so are (\binom{3}{2}) and (\binom{3}{1}), but these make the identity nicer), and we can spot the general pattern:

\[
\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{0}{m} + \binom{1}{m} + \cdots + \binom{n}{m} = \binom{n+1}{m+1}, \text{ integers } m, n \geq 0. \tag{5.10}
\]

This identity, which we call summation on the upper index, expresses a binomial coefficient as the sum of others whose lower indices are constant. In this case the sum needs the lower limit \(k \geq 0\), because the terms with \(k < 0\) aren’t zero. Also, \(m\) and \(n\) can’t in general be negative.

Identity (5.10) has an interesting combinatorial interpretation. If we want to choose \(m + 1\) tickets from a set of \(n + 1\) tickets numbered 0 through \(n\), there are \(\binom{n+1}{m+1}\) ways to do this when the largest ticket selected is number \(k\).

We can prove both (5.9) and (5.10) by induction using the addition formula, but we can also prove them from each other. For example, let’s prove (5.9) from (5.10); our proof will illustrate some common binomial coefficient manipulations. Our general plan will be to massage the left side \(\sum (\binom{k}{i})\) of (5.9) so that it looks like the left side \(\sum (\binom{k}{m})\) of (5.10); then we’ll invoke that identity, replacing the sum by a single binomial coefficient; finally we’ll transform that coefficient into the right side of (5.9).

We can assume for convenience that \(r\) and \(n\) are nonnegative integers; the general case of (5.9) follows from this special case, by the polynomial argument. Let’s write \(m\) instead of \(r\), so that this variable looks more like a nonnegative integer. The plan can now be carried out systematically as