Proof of Theorem 1. It is straightforward that

(i) \( H(G_1 \times G_2) \cong H(G_1) \otimes H(G_2) \),

(ii) \( H(G_1 \times G_2, K_1 \times K_2) \cong H(G_1, K_1) \otimes H(G_2, K_2) \) and

(iii) \( (W_1 \otimes W_2)^{K_1 \times K_2} \cong W_1^{K_1} \otimes W_2^{K_2} \)

for every pair of compact open subgroups \( K_i \) of \( G_i \) and every pair of smooth \( G_i \)-modules \( W_i \).

Assertion (1.1) follows from (iii) and the irreducibility criterion.

Conversely, let \( W \) be an admissible irreducible \( G \)-module. Let \( K = K_1 \times K_2 \), where \( K_i \) is a compact open subgroup of \( G_i \), \( i = 1, 2 \), be such that \( W^K \neq 0 \). The space \( W^K \) is finite dimensional, so by the corollary on p. 94 of [2] there exist irreducible \( H(G_i, K_i) \)-modules \( W_i^{K_i} \) and an \( H(G, K) \) isomorphism \( a_K \) from \( W^K \) to \( W_1^{K_1} \otimes W_2^{K_2} \). Similar remarks apply to every open subgroup \( K' = K_1' \times K_2' \) of \( K \). There exist \( H(G_i, K_i) \)-maps \( b_i = b_i(K, K') : W_i^{K_i} \rightarrow W_i^{K_i'} \) such that the following diagram is commutative.

\[
\begin{array}{c}
W^K & \xrightarrow{a_K} & W_1^{K_1} \otimes W_2^{K_2} \\
\downarrow \text{incl.} & & \downarrow \text{incl.} \\
W'^K & \xrightarrow{a_{K'}} & W_1^{K_1'} \otimes W_2^{K_2'} \\
\end{array}
\]

Moreover, the maps \( b_i(K, K') \) can be chosen for every pair of compact open subgroups \( K, K' \) of this type in such a way as to form an inductive system. Then \( W \cong W_1 \otimes W_2 \), where \( W_i = \text{ind lim}_K W_i^{K_i} \), and \( W \) is an admissible irreducible representation of \( G_i \), \( i = 1, 2 \).

The class of \( W \) is determined by that of \( W_i \) for the restriction of \( W \) to \( G_i \) is \( W_i \)-isotypic. \( \square \)

An analysis of the proof of Theorem 1 reveals that the groups \( G \) and \( G_i \) enter only through their Hecke algebras. This leads one to define an idempotented algebra \( (A, E) \) to be an algebra \( A \) with a directed family of idempotents \( E \) such that \( A = \bigcup e A e \). An admissible module \( W \) for \( (A, E) \) is an \( A \)-module \( W \) which is nondegenerate in the sense that \( A W = W \) and is such that \( \dim e W \) is finite for all \( e \in E \). The tensor product of two idempotented algebras is naturally idempotented. The proof of Theorem 1 is readily adapted to establish a similar theorem about the admissible irreducible modules of the tensor product of two idempotented algebras.

2. The study of the representations of adelic groups, which are infinite restricted products of groups, requires the notion of restricted tensor product of vector spaces which was introduced in [4].

Let \( \{ W_v | v \in V \} \) be a family of vector spaces. Let \( V_0 \) be a finite subset of \( V \). For each \( v \in V \setminus V_0 \), let \( x_v \) be a nonzero vector in \( W_v \). For each finite subset \( S \) of \( V \) containing \( V_0 \), let \( V_S = \bigotimes_{v \in S} W_v \); and if \( S \subset S' \), let \( f_S : W_S \rightarrow W_{S'} \) be defined by \( \bigotimes_{v \in S} w_v \mapsto \bigotimes_{v \in S} w_v \otimes_{v \in S' \setminus S} x_v \). Then \( W = \bigotimes_{v \in V} W_v \), the restricted tensor product of the \( W_v \) with respect to the \( x_v \), is defined by \( W = \text{ind lim}_S W_S \). The space \( W \) is spanned by elements written in the form \( w = \bigotimes w_v \), where \( w_v = x_v \) for almost all \( v \in V \).

The ordinary constructions with finite tensor products extend easily to restricted tensor products.
1. Given linear maps $B_v: W_v \to W_v$ such that $B_v x_v = x_v$ for almost all $v \in V$, then one can define $B = \bigotimes B_v : W \to W$ by $B(\bigotimes w_v) = \bigotimes B_v w_v$.

2. Given a family of algebras $\{A_v\}_{v \in V}$ and given nonzero idempotents $e_v \in A_v$ for almost all $v$, then $A = \bigotimes e_v A_v$ is an algebra in the obvious way.

3. If $W_v$ is an $A_v$-module for each $v \in V$ such that $e_v : x_v = x_v$ for almost all $v$, then $\bigotimes e_v W_v$ is an $A$-module. The isomorphism class of $W$ depends on $\{x_v\}$. However, if $\{x_v\}$ is another collection of nonzero vectors such that $x_v$ and $x_v'$ lie on the same line in $W_v$ for almost all $v$, then the $A$-modules $\bigotimes e_v W_v$ and $\bigotimes e_v' W_v$ are isomorphic.

Example 1. The polynomial ring in an infinite number of variables $C[X_1, X_2, \ldots]$ is isomorphic to $\bigotimes e_v C[X_v]$, where $e_v$ is the identity element of $C[X_v]$.

Example 2. Let $G = \prod_{K_v} G_v$ be the restricted product of locally compact totally disconnected groups $G_v$, restricted with respect to the compact open subgroups $K_v$. Then $G$ itself is locally compact and totally disconnected, and $H(G)$ is isomorphic to $\bigotimes e_v H(G_v)$.

For each $v \in V$ let $W_v$ be an admissible $G_v$-module. Assume that $\dim W^K_v = 1$ for almost all $v$. Choosing for almost all $v$ a nonzero vector $x_v \in W^K_v$, we may form the $G$-module $W = \bigotimes x_v W_v$. The isomorphism class of $W$ is in fact independent of the choice of $x_v \in W^K_v$ and will be called the tensor product of the representations $W_v$. One sees that $W$ is admissible, and that it is irreducible if and only if each $W_v$ is. The admissible irreducible representations of $G$ isomorphic to ones constructed in this way are said to be factorizable.

Theorem 2. Suppose that $H(G_v, K_v)$ is commutative for almost all $v$. Then every admissible irreducible representation $W$ of $G$ is factorizable, $W \cong \bigotimes W_v$. The isomorphism classes of the factors $W_v$ are determined by that of $W$. For almost all $v$, $\dim W^K_v = 1$.

Proof. One first factorizes the finite dimensional spaces $W^K_v$ for compact open subgroups $K_v = \prod K_v$ of $G$, then continues as in the proof of Theorem 1.

3. Let $G$ be a connected reductive algebraic group over a global field $F$. Let $A$ be the adele ring of $F$, and let $V$ be the set of places of $F$. The adelic group $G(A)$ is isomorphic to a restricted product $\prod_K^* G(F_v)$, where the subgroups $K_v$ are defined for all finite $v$ and are certain maximal compact subgroups of $G(F_v)$. For almost all finite $v \in V$, $G(F_v)$ is a quasi-split group over $F_v$, and $K_v$ is a special maximal compact subgroup. For these places $v$, $H(G(F_v), K_v)$ is commutative. See [5]. So the function field case of the following theorem, whose meaning has yet to be explained in the number field case, is a special case of Theorem 2.

Theorem 3. Every admissible irreducible representation of $G(A)$ is factorizable. The factors are unique up to equivalence.

Let $F$ be a number field. Then the class of admissible representations of $G(A)$ has yet to be defined. For each archimedean place $v \in V$, let $K_v$ be a maximal compact subgroup of $G(F_v)$, and let $g_v$ be the real Lie algebra of $G(F_v)$. Let $K_\infty = \prod_{v \text{ arch}} K_v$, $K = \prod_{v \text{ all}} K_v$, and $G_\infty = \prod_{v \text{ arch}} G(F_v)$. Let $g_\infty$ be the real Lie algebra of $G_\infty$. Let $A_f$ be the ring of finite adeles of $F$. 