Table 164 Pascal’s triangle, extended upward.

<table>
<thead>
<tr>
<th>n</th>
<th>(n)</th>
<th>(n)</th>
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<th>(n)</th>
<th>(n)</th>
<th>(n)</th>
<th>(n)</th>
<th>(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-4</td>
<td>1</td>
<td>-4</td>
<td>10</td>
<td>-20</td>
<td>35</td>
<td>-56</td>
<td>84</td>
<td>-120</td>
<td>165</td>
<td>-220</td>
</tr>
<tr>
<td>-3</td>
<td>1</td>
<td>-3</td>
<td>6</td>
<td>-10</td>
<td>15</td>
<td>-21</td>
<td>28</td>
<td>-36</td>
<td>45</td>
<td>-55</td>
</tr>
<tr>
<td>-2</td>
<td>1</td>
<td>-2</td>
<td>3</td>
<td>-4</td>
<td>5</td>
<td>-6</td>
<td>7</td>
<td>-8</td>
<td>9</td>
<td>-10</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

All these numbers are familiar. Indeed, the rows and columns of Table 164 appear as columns in Table 155 (but minus the minus signs). So there must be a connection between the values of \( \binom{n}{k} \) for negative \( n \) and the values for positive \( n \). The general rule is

\[
\binom{r}{k} = (-1)^k \binom{k-r-1}{k}, \quad \text{integer } k;
\]

(5.14)

it is easily proved, since

\[
r^k = r(r-1)\ldots(r-k+1) = (-1)^k(-r)(1-r)\ldots(k-1-r) = (-1)^k(k-r-1)^k
\]

when \( k \geq 0 \), and both sides are zero when \( k < 0 \).

Identity (5.14) is particularly valuable because it holds without any restriction. (Of course, the lower index must be an integer so that the binomial coefficients are defined.) The transformation in (5.14) is called negating the upper index, or “upper negation!”

But how can we remember this important formula? The other identities we’ve seen—symmetry, absorption, addition, etc.—are pretty simple, but this one looks rather messy. Still, there’s a mnemonic that’s not too bad: To negate the upper index, we begin by writing down \((-1)^k\), where \( k \) is the lower index. (The lower index doesn’t change.) Then we immediately write \( k \) again, twice, in both lower and upper index positions. Then we negate the original upper index by subtracting it from the new upper index. And we complete the job by subtracting 1 more (always subtracting, not adding, because this is a negation process).

Let’s negate the upper index twice in succession, for practice. We get

\[
\binom{r}{k} = (-1)^2k \binom{k-r-1}{k} = (-1)^2k \binom{k-(k-r-1)-1}{k} = \binom{r}{k},
\]

You call this a mnemonic? I’d call it pneumatic—full of air. It does help me remember, though. (Now is a good time to do warmup exercise 4.)