not rows. Curiously, however, there is a way to partially sum the row elements if they have been multiplied by their distance from the center:

\[
\sum_{k \leq m} \binom{r}{k} \left( \frac{r}{2} - k \right) = \frac{m + 1}{2} \binom{r}{m + 1}, \quad \text{integer } m. \tag{5.18}
\]

(This formula is easily verified by induction on \(m\).) The relation between these partial sums with and without the factor of \((r/2 - k)\) in the summand is analogous to the relation between the integrals

\[
\int_{-\infty}^{\alpha} xe^{-x^2} \, dx = -\frac{1}{2} \sqrt{\pi} \text{erf} \alpha \quad \text{and} \quad \int_{-\infty}^{\alpha} e^{-x^2} \, dx.
\]

The apparently more complicated integral on the left, with the factor of \(x\), has a closed form, while the simpler-looking integral on the right, without the factor, has none. Appearances can be deceiving.

At the end of this chapter, we’ll study a method by which it’s possible to determine whether or not there is a closed form for the partial sums of a given series involving binomial coefficients, in a fairly general setting. This method is capable of discovering identities (5.16) and (5.18), and it also will tell us that (5.17) is a dead end.

Partial sums of the binomial series lead to a curious relationship of another kind:

\[
\sum_{k \leq m} \binom{m + r}{k} x^k y^{m-k} = \sum_{k \leq m} \binom{-r}{k} (-x)^k (x + y)^{m-k}, \quad \text{integer } m. \tag{5.19}
\]

This identity isn’t hard to prove by induction: Both sides are zero when \(m < 0\) and 1 when \(m = 0\). If we let \(S_m\) stand for the sum on the left, we can apply the addition formula (5.8) and show easily that

\[
S_m = \sum_{k \leq m} \binom{m - 1 + r}{k} x^k y^{m-k} + \sum_{k \leq m} \binom{m - 1 + r}{k - 1} x^k y^{m-k};
\]

and

\[
\sum_{k \leq m} \binom{m - 1}{k} x^k y^{m-k} = yS_{m-1} + \binom{m - 1 + r}{m} x^m,
\]

\[
\sum_{k \leq m} \binom{m - 1 + r}{k - 1} x^k y^{m-k} = xS_{m-1},
\]

when \(m > 0\). Hence

\[
S_m = (x + y)S_{m-1} + \binom{-r}{m} (-x)^m.
\]