with infinitely differentiable coefficients $b_{k; \alpha; j}$, and $C(x, t, \partial_x, \partial_t)$ is a matrix of tangential differential operators $C_{k,j}(x, t, \partial_x, \partial_t)$ of order $\leq \mu_k + \tau_j$ on $\partial C$ with smooth coefficients $c_{k,j; \alpha; s}$. We assume again that $\text{ord } B_k < 2m$ for $k = 1, \ldots, m + J$. Then the vector $B$ admits the representation

\begin{equation}
B(x, t, \partial_x, \partial_t) u \big|_{\partial C} = Q(x, t, \partial_x, \partial_t) \cdot Du \big|_{\partial C},
\end{equation}

where $Q$ is a $(m + J) \times J$-matrix of tangential differential operators $Q_{k,j}(x, t, \partial_x, \partial_t)$, ord $Q_{k,j} \leq \mu_k + 1 - j$, $Q_{k,j} \equiv 0$ if $\mu_k + 1 - j < 0$.

Moreover, we suppose that the coefficients of $L, B,$ and $C$ stabilize for $t \to \pm \infty$, i.e., there exist smooth functions $a_{\alpha; j}^{(0)}, b_{k; \alpha; j}^{(0)}, c_{k,j; \alpha; s}^{(0)}$ on $\overline{\Omega}$ and in a neighbourhood of $\partial \Omega$, respectively, such that

\begin{align*}
&\partial_{\alpha}^{\mu} (a_{\alpha; j}(x, t) - a_{\alpha; j}^{(0)}(x)) \to 0 \quad \text{as } t \to \pm \infty, \\
&\partial_{\alpha}^{\mu} (b_{k; \alpha; j}(x, t) - b_{k; \alpha; j}^{(0)}(x)) \to 0 \quad \text{as } t \to \pm \infty \\
&\partial_{\alpha}^{\mu} (c_{k,j; \alpha; s}(x, t) - c_{k,j; \alpha; s}^{(0)}(x)) \to 0 \quad \text{as } t \to \pm \infty
\end{align*}

uniformly with respect to $x$ for all nonnegative integer $\mu$ and all multi-indices $\gamma$.

We denote the operator of the problem (5.5.1), (5.5.2) by $A(t, \partial_t)$, while $A_0(\partial_t)$ denotes the operator of the model problem

\begin{align}
&L^{(0)}(x, \partial_x, \partial_t) u = f \quad \text{in } C, \\
&B^{(0)}(x, \partial_x, \partial_t) u + C^{(0)}(x, \partial_x, \partial_t) u = g \quad \text{on } \partial C
\end{align}

which arises from (5.5.1), (5.5.2) if we replace the coefficients $a_{\alpha; j}(x, t), b_{k; \alpha; j}(x, t), c_{k,j; \alpha; s}(x, t)$ by $a_{\alpha; j}^{(0)}(x, t), b_{k; \alpha; j}^{(0)}(x, t),$ and $c_{k,j; \alpha; s}^{(0)}(x, t)$, respectively. Both operators $A(t, \partial_t)$ and $A_0(\partial_t)$ continuously map the space (5.2.16) into (5.2.17) for arbitrary integer $l \geq 2m$ and real $\beta$. Furthermore, as an immediate consequence of the stabilization condition, we get the following lemma.

\textbf{Lemma 5.5.1.} Suppose that the coefficients of $L$ stabilize for $t \to \pm \infty$. Then there exists a constant $c_T$ such that

$$\|L(x, t, \partial_x, \partial_t) - L^{(0)}(x, \partial_x, \partial_t)\|_{W_{2,s}^l(\Omega)} \leq c_T \|u\|_{W_{2,s}^l(\Omega)}$$

for all $u \in W_{2,s}^l(\Omega)$ equal to zero in $\Omega \times (-T, +T)$, $l \geq 2m$. The factor $c_T$ tends to zero as $T \to +\infty$.

Analogous assertions are valid for the operators $B_k$ and $C_{k,j}$.

\textbf{5.5.2. Extension of the operator corresponding to the boundary value problem.} Analogously to the case when $L, B,$ and $C$ are model operators, the following Green formula is valid for all $u, v \in C_{0}^{\infty}(\overline{\Delta}), u \in C_{0}^{\infty}(\partial C)^{J}$, $v \in C_{0}^{\infty}(\partial C)^{m+J} :$

\begin{equation}
(L(t, \partial_t)u, v)_{C} + (B(t, \partial_t)u + C(t, \partial_t)u, v)_{\partial C}
= (u, L^+(t, \partial_t) v)_{C} + (Du, P(t, \partial_t) u + Q^+(t, \partial_t) u)_{\partial C} + (u, C^+(t, \partial_t) u)_{\partial C}
\end{equation}

(for the sake of brevity, we have omitted the arguments $x, \partial_x$ in the differential operators). Obviously, the coefficients of the formally adjoint operators $L^+(t, -\partial_t), Q^+(t, -\partial_t), C^+(t, -\partial_t)$ to $L(t, \partial_t), Q(t, \partial_t), C(t, \partial_t)$ stabilize at infinity. Furthermore, it can be easily shown that the coefficients of $P(t, -\partial_t)$ stabilize at infinity. This follows from the explicit formula (2.3.12) for the operator $P$ in the Green formula for the half-space. In the same way as it was carried out for the operator $A_0(\partial_t)$ (see Lemma 5.3.3, Theorem 5.3.1), the operator $A(t, \partial_t)$ can be extended to the space (5.3.11) with arbitrary integer $l$. This leads to the following results.
**Lemma 5.5.2.** The operator
\[ \mathcal{V}_{2,\beta}^{2m,2m}(C) \ni (u, Du|_{\partial C}) \rightarrow Lu = f \in \mathcal{W}_{2,\beta}^0(C) \]
can be uniquely extended to a continuous operator
\[ \mathcal{W}_{2,\beta}^l,2m(C) \ni (u, \phi) \rightarrow f \in \mathcal{W}_{2,-\beta}^{2m-l}(C)^*, \quad l < 2m. \]

The functional \( f = L(u, \phi) \in \mathcal{W}_{2,-\beta}^{2m-l}(C)^* \) in (5.5.7) is given by the same formulas as in the case of \( t \)-independent coefficients (cf. Lemma 5.3.3). In the case \( l \leq 0 \) we have
\[ (f, v) = (u, L^+(t, -\partial_t)v)_{\partial C} + \sum_{j=1}^{2m} (\phi_j, P_j(t, -\partial_t)v)_{\partial C} \]
for \( v \in \mathcal{W}_{2,-\beta}^{2m-l}(C) \), while in the case \( 0 < l < 2m \) the formula (5.3.15) is valid, where \( L_{\alpha,j}, P_j \) and \( P_{l,j} \) are differential operators with \( t \)-dependent coefficients stabilizing at infinity.

**Theorem 5.5.1.** Suppose that the coefficients of \( L, B, \) and \( C \) stabilize at infinity. Then the operator
\[ \mathcal{W}_{2,\beta}^l,2m(C) \times \mathcal{W}_{2,\beta}^{l+\tau-1/2}(\partial C) \ni (u, Du|_{\partial C}, y) \rightarrow (Lu, Bu|_{\partial C} + Cu) \in \mathcal{W}_{2,\beta}^{l-2m}(C) \times \mathcal{W}_{2,\beta}^{-\mu-1/2}(\partial C) \]
with \( l \geq 2m \) can be uniquely extended to a linear and continuous operator
\[ \mathfrak{A}(t, \partial_t) : \mathcal{W}_{2,\beta}^l,2m(C) \times \mathcal{W}_{2,\beta}^{l+\tau-1/2}(\partial C) \rightarrow \mathcal{W}_{2,\beta}^{l-2m,0}(C) \times \mathcal{W}_{2,\beta}^{-\mu-1/2}(\partial C) \]
with \( l < 2m \). This extension has the form
\[ (u, \phi, y) \rightarrow (L(u, \phi), Q\phi + Cy), \]
where \( L \) is the operator (5.5.7) and \( Q \) is the matrix in (5.5.3).

Due to the stabilization condition on the coefficients of the operators \( L, B, \) and \( C \) we can generalize the regularity assertion of Lemma 5.3.5.

**Lemma 5.5.3.** Suppose that the coefficients of \( L, B, C \) stabilize at infinity and the boundary value problem (5.5.1), (5.5.2) is elliptic. If \( (u, \phi, y) \in \mathcal{W}_{2,\beta}^l,2m(C) \times \mathcal{W}_{2,\beta}^{l+\tau-1/2}(\partial C) \) is a solution of the equation \( \mathfrak{A}(t, \partial_t)(u, \phi, y) = (f, g) \), where \( f \in \mathcal{W}_{2,\beta}^{l-2m+1,0}(C) \) and \( g \in \mathcal{W}_{2,\beta}^{-\mu+1/2}(\partial C) \), then \( (u, \phi, y) \in \mathcal{W}_{2,\beta}^{l+1,2m}(C) \times \mathcal{W}_{2,\beta}^{l+\tau+1/2}(\partial C) \). Furthermore, the inequality
\[ \|(u, \phi, y)\|_{l+1, \beta} \leq c \left( \|f\|_{\mathcal{W}_{2,\beta}^{l-2m+1,0}(C)} + \|g\|_{\mathcal{W}_{2,\beta}^{-\mu+1/2}(\partial C)} + \|(u, \phi, y)\|_{l, \beta} \right) \]
is satisfied with a constant \( c \) independent of \( (u, \phi, y) \). Here \( \| \cdot \|_{l, \beta} \) denotes the norm in \( \mathcal{W}_{2,\beta}^{l,2m}(C) \times \mathcal{W}_{2,\beta}^{l+\tau-1/2}(\partial C) \).

**Proof:** Let \( (u, \phi, y) \) be an element of the space \( \mathcal{W}_{2,\beta}^{l,2m}(C) \times \mathcal{W}_{2,\beta}^{l+\tau-1/2}(\partial C) \) and let \( \zeta_k, \eta_k \) be the same functions as in the proof of Lemma 5.3.4. From Lemma 3.2.4 it follows that \( \zeta_k(u, \phi, y) \in \mathcal{W}_{2,\beta}^{l+1,2m}(C) \times \mathcal{W}_{2,\beta}^{l+\tau+1/2}(\partial C) \) and
\[ \|(\zeta_k(u, \phi, y))\|_{l+1, \beta} \leq c_k \left( \|\zeta_k f\|_{\mathcal{W}_{2,\beta}^{l-2m+1,0}(C)} + \|\zeta_k g\|_{\mathcal{W}_{2,\beta}^{-\mu+1/2}(\partial C)} + \|\eta_k(u, \phi, y)\|_{l, \beta} \right) \]