Equation (5.21) holds primarily because of cancellation between \( m! \)'s in the factorial representations of \( \binom{r}{m} \) and \( \binom{T}{k} \). If all variables are integers and \( r \geq m \geq k \geq 0 \), we have

\[
\binom{r}{m} \binom{m}{k} = \frac{r!}{m!(r-m)!} \frac{m!}{k!(m-k)!} = \frac{r!}{k!(r-k)!} \frac{(r-k)!}{(m-k)!(r-m)!} = \binom{r}{k} \binom{r-k}{m-k}.
\]

That was easy. Furthermore, if \( m < k \) or \( k < 0 \), both sides of (5.21) are zero; so the identity holds for all integers \( m \) and \( k \). Finally, the polynomial argument extends its validity to all real \( r \).

A binomial coefficient \( \binom{r}{k} = \frac{r!}{k!(r-k)!} \) can be written in the form \( (a + b)!/a! b! \) after a suitable renaming of variables. Similarly, the quantity in the middle of the derivation above, \( r!/k!(m-k)!(r-m)! \), can be written in the form \( (a + b + c)!/a! b! c! \). This is a "trinomial coefficient," which arises in the "trinomial theorem":

\[
(x + y + z)^n = \sum \binom{a + b + c}{a, b, c} x^a y^b z^c.
\]

So \( \binom{r}{k} \) is really a trinomial coefficient in disguise. Trinomial coefficients pop up occasionally in applications, and we can conveniently write them as

\[
\binom{a + b + c}{a, b, c} = \frac{(a + b + c)!}{a! b! c!}
\]

in order to emphasize the symmetry present.

Binomial and trinomial coefficients generalize to multinomial coefficients, which are always expressible as products of binomial coefficients:

\[
\binom{a_1 + a_2 + \ldots + a_n}{a_1, a_2, \ldots, a_m} = \frac{(a_1 + a_2 + \ldots + a_m)!}{a_1! a_2! \ldots a_m!} = \binom{a_1 + a_2 + \ldots + a_n}{a_2 + \ldots + a_n, a_1} \ldots \binom{a_m - 1 + a_m}{a_m}.
\]

Therefore, when we run across such a beastie, our standard techniques apply.