Equation (5.21) holds primarily because of cancellation between $m!$’s in the factorial representations of $\binom{r}{m}$ and $\binom{r}{k}$. If all variables are integers and $r \geq m \geq k \geq 0$, we have

\[
\binom{r}{m} \binom{m}{k} = \frac{r!}{m!(r-m)!} \cdot \frac{m!}{k!(m-k)!} = \frac{r!}{k!(r-k)!} \cdot \frac{(r-k)!}{(m-k)!(r-m)!} = \binom{r}{k} \binom{r-k}{m-k}.
\]

That was easy. Furthermore, if $m < k$ or $k < 0$, both sides of (5.21) are zero; so the identity holds for all integers $m$ and $k$. Finally, the polynomial argument extends its validity to all real $r$.

A binomial coefficient $\binom{r}{k} = r!/(r-k)!$ can be written in the form $(a + b)!/a!b!$ after a suitable renaming of variables. Similarly, the quantity in the middle of the derivation above, $r!/k!(m-k)!(r-m)!$, can be written in the form $(a + b + c)!/a!b!c!$. This is a “trinomial coefficient,” which arises in the “trinomial theorem”:

\[
(x + y + z)^n = \sum_{\substack{a, b, c \in \mathbb{N} \\ a + b + c = n}} \frac{(a + b + c)!}{a!b!c!} x^a y^b z^c.
\]

So $\binom{r}{m}(\binom{m}{k})$ is really a trinomial coefficient in disguise. Trinomial coefficients pop up occasionally in applications, and we can conveniently write them as

\[
\binom{a + b + c}{a, b, c} = \frac{(a + b + c)!}{a!b!c!}
\]

in order to emphasize the symmetry present.

Binomial and trinomial coefficients generalize to multinomial coefficients, which are always expressible as products of binomial coefficients:

\[
\binom{a_1 + a_2 + \ldots + a_n}{a_1, a_2, \ldots, a_m} = \frac{(a_1 + a_2 + \ldots + a_n)!}{a_1! a_2! \ldots a_m!} = \binom{a_1 + a_2 + \ldots + a_n}{a_2 + \ldots + a_n, a_m} \cdots \binom{a_{m-1} + a_m}{a_m}.
\]

Therefore, when we run across such a beastie, our standard techniques apply.