5.1 BASIC IDENTITIES

Table 169 Sums of products of binomial coefficients.

\[ \sum_k \binom{r}{m+k} \binom{s}{n-k} = \binom{r+s}{m+n}, \quad \text{integers } m, n. \quad (5.22) \]

\[ \sum_k \binom{l}{m+k} \binom{s}{n+k} = \binom{l+s}{1+m+n}, \quad \text{integer } l \geq 0, \text{ integers } m, n. \quad (5.23) \]

\[ \sum_k \binom{l}{m+k} \binom{s+k}{n} (-1)^k \approx (-1)^{1+m} \binom{s-m}{n-t}, \quad \text{integer } l \geq 0, \text{ integers } m, n. \quad (5.24) \]

\[ \sum_{k \leq 1} \binom{l-k}{m} \binom{s}{k-n} (-1)^k = (-1)^{1+m} \binom{s-m-1}{l-m-n}, \quad \text{integers } l, m, n \geq 0. \quad (5.25) \]

\[ \sum_{0 \leq k \leq 1} \binom{l-k}{m} \binom{q+k}{n} = \binom{l+q+1}{m+n+1}, \quad \text{integers } l, m \geq 0, \text{ integers } n \geq q \geq 0. \quad (5.26) \]

Now we come to Table 169, which lists identities that are among the most important of our standard techniques. These are the ones we rely on when struggling with a sum involving a product of two binomial coefficients. Each of these identities is a sum over \( k \), with one appearance of \( k \) in each binomial coefficient; there also are four nearly independent parameters, called \( m, n, r, t \), etc., one in each index position. Different cases arise depending on whether \( k \) appears in the upper or lower index, and on whether it appears with a plus or minus sign. Sometimes there’s an additional factor of \((-1)^k\), which is needed to make the terms summable in closed form.

Table 169 is far too complicated to memorize in full; it is intended only for reference. But the first identity in this table is by far the most memorable, and it should be remembered. It states that the sum (over all integers \( k \)) of the product of two binomial coefficients, in which the upper indices are constant and the lower indices have a constant sum for all \( k \), is the binomial coefficient obtained by summing both lower and upper indices. This identity is known as Vandermonde’s convolution, because Alexandre Vandermonde wrote a significant paper about it in the late 1700s [293]; it was, however, known to Chu Shih-Chieh in China as early as 1303. All of the other identities in Table 169 can be obtained from Vandermonde’s convolution by doing things like negating upper indices or applying the symmetry law, etc., with care; therefore Vandermonde’s convolution is the most basic of all.

We can prove Vandermonde’s convolution by giving it a nice combinatorial interpretation. If we replace \( k \) by \( k - m \) and \( n \) by \( n - m \), we can assume...