Chapter 3. Second Order Linear Equations

In each of Problems 1 through 10 find the general solution of the given differential equation.

**EXAMPLE 3**

Given that \( y_1(t) = t^{-1} \) is a solution of

\[
2t^2y'' + 3ty' - y = 0, \quad t > 0,
\]

find a second linearly independent solution.

We set \( y = v(t)t^{-1} \); then

\[
y' = v't^{-1} - vt^{-2}, \quad y'' = v''t^{-1} - 2v't^{-2} + 2vt^{-3}.
\]

Substituting for \( y, y', \) and \( y'' \) in Eq. (31) and collecting terms, we obtain

\[
2t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3}) + 3t(v't^{-1} - vt^{-2}) - v't^{-1}
\]

\[
= 2tv'' + (-4 + 3)v' + (4t^{-1} - 3t^{-1} - t^{-1})v
\]

\[
= 2tv'' - v' = 0.
\]

Note that the coefficient of \( v \) is zero, as it should be; this provides a useful check on our algebra.

Separating the variables in Eq. (32) and solving for \( v'(t) \), we find that

\[
v'(t) = ct^{1/2};
\]

then

\[
v(t) = \frac{2}{3}ct^{3/2} + k.
\]

It follows that

\[
y = \frac{2}{3}ct^{1/2} + kt^{-1},
\]

where \( c \) and \( k \) are arbitrary constants. The second term on the right side of Eq. (33) is a multiple of \( y_1(t) \) and can be dropped, but the first term provides a new independent solution. Neglecting the arbitrary multiplicative constant, we have \( y_2(t) = t^{1/2} \).

**PROBLEMS**

In each of Problems 1 through 10 find the general solution of the given differential equation.

1. \( y'' - 3y' + y = 0 \)
2. \( 9y'' + 6y' + y = 0 \)
3. \( 4y'' - 4y' - 3y = 0 \)
4. \( 4y'' + 12y' + 9y = 0 \)
5. \( y'' - 2y' + 10y = 0 \)
6. \( y'' - 6y' + 9y = 0 \)
7. \( 4y'' + 17y' + 4y = 0 \)
8. \( 16y'' + 24y' + 9y = 0 \)
9. \( 25y'' - 20y' + 4y = 0 \)
10. \( 2y'' + 2y' + y = 0 \)

In each of Problems 11 through 14 solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing \( t \).

11. \( 9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1 \)
12. \( y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2 \)
13. \( 9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2 \)
14. \( y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1 \)