1.6. *Growth condition*. (1) Let $A$ be the identity component of a maximal $Q$-split torus $S$ of $G$, and $\Phi$ the system of $Q$-roots of $G$ with respect to $S$. Fix an ordering on $\Phi$ and let $\Delta$ be the set of simple roots. Given $t > 0$ let

$$A_t = \{ a \in A : |\alpha(a)| \geq t, (a \in \Delta) \}.$$  

Let $f$ be a function satisfying 1.3(a), (b), (c). Then the growth condition (d) is equivalent to:

(d') Given a compact set $R \subset G(R)$, and $t > 0$, there exist a constant $C > 0$ and a positive integer $m$ such that

$$|f(x \cdot a)| \leq C \cdot |\alpha(a)|^m, \quad \text{for all } a \in A_t, \alpha \in \Delta, x \in R.$$  

This follows from reduction theory [11, §2]. More precisely, let $G'$ be the derived group of $G$. Then $A$ is the direct product of $Z(R)^{e}$ and $A' = A \cap G'(R)$. For a function satisfying 1.3(a), (b), the growth condition (d') is equivalent to (d) for $a \in A_t'$; but says nothing for $a \in Z(R)^{\circ}$. However condition (c) implies that $f$ depends polynomially on $z \in Z(R)$, and this takes care of the growth condition on $Z(R)$.

(2) Assume $f$ satisfies 1.3(a), (b), (c) and

$$f(x \cdot z) = \chi(z)f(x) \quad (z \in Z(R), x \in G(R))$$  

where $\chi$ is a character of $Z(R)/(Z(R) \cap I')$. Then $|f|$ is a function on $Z(R) \cdot I' \cap G(R)$. If $|f| \in L^p(Z(R) \cap G(R))$ for some $p \geq 1$, then $f$ is slowly increasing, hence is an automorphic form. In view of the fact that $Z(R)/(Z(R) \cap G(R))$ has finite invariant volume, it suffices to prove this for $p = 1$. In that case, it follows from the corollary to Lemma 9 in [11], and from the existence of a $K$-invariant function $\alpha \in C^\infty_c(G(R))$ such that $f = f \ast \alpha$ (a well-known property of $K$-finite and $Z(q)$-finite elements in a differentiable representation of $G(R)$, which follows from 2.1 below).

1.7. **Theorem** [11, Theorem 1]. The space $\mathcal{A}(\Gamma, \xi, J, K)$ is finite dimensional.

This theorem is due to Harish-Chandra. Actually the proof given in [11] is for semisimple groups, but the extension to reductive groups is easy. In fact, it is implicitly done in the induction argument of [11] to prove the theorem. For another proof, see [13, Lemma 3.5]. At any rate, it is customary to fix a quasi-character $\chi$ of $Z(R)/(\Gamma' \cap Z(R))$ and consider the space $\mathcal{A}(\Gamma, \xi, J, K)_\chi$ of elements in $\mathcal{A}(\Gamma, \xi, J, K)$ which satisfy 1.6(3). For those, the reduction to the semisimple case is immediate. Note that since the identity component $Z(R)^{\circ}$ of $Z(R)$ (sometimes called the split component of $G(R)$) has finite index in $Z(R)$ and $Z(R)^{\circ} \cap \Gamma' = \{1\}$, it is substantially equivalent to require 1.6(3) for an arbitrary quasi-character of $Z(R)^{\circ}$.

The space $\mathcal{A}(\Gamma, \xi, J, K)$ is acted upon by the center $C(G(R))$ of $G(R)$, by left or right translations. Since it is finite dimensional, we see that any automorphic form is $C(G(R))$-finite.

1.8. **Cusp forms.** A continuous (resp. measurable) function on $G(R)$ is cuspidal if

$$\int_{(\Gamma' \cap N(R)) \backslash N(R)} f(n \cdot x) \, dn = 0,$$

for all (resp. almost all) $x$ in $G(R)$, where $N$ is the unipotent radical of any proper
parabolic $Q$-subgroup of $G$. It suffices in fact to require this for any proper maximal parabolic $Q$-subgroup [11, Lemma 3].

A cusp form is a cuspidal automorphic form. We let $\mathcal{A}(I', \xi, J, K)$ be the space of cusp forms in $\mathcal{A}(I', \xi, J, K)$.

Let $f$ be a smooth function on $G(\mathbf{R})$ satisfying the conditions (a), (b), (c) of 1.3. Assume that $f$ is cuspidal and that there exists a character $\chi$ of $Z(\mathbf{R})$ such that 1.6(3) is satisfied. Then the following conditions are equivalent:

(i) $f$ is slowly increasing, i.e., $f$ is a cusp form;

(ii) $f$ is bounded;

(iii) $|f|$ is square-integrable modulo $Z(\mathbf{R}) \cdot I'$
(cf. [11, §4]). In fact, one has much more: $|f|$ decreases very fast to zero at infinity on $Z(\mathbf{R})\Gamma' \setminus G(\mathbf{R})$, so that if $g$ is any automorphic form satisfying 1.6(3), then $|f \cdot g|$ is integrable on $Z(\mathbf{R}) \cdot I' \setminus G(\mathbf{R})$ (loc. cit.).

The space $\mathcal{A}(I', \xi, J, K)$ of the functions in $\mathcal{A}(I', \xi, J, K)$ satisfying 1.6(3) may then be viewed as a closed subspace of bounded functions in the space $L^2(I' \setminus G(\mathbf{R}))$ of functions on $I' \setminus G(\mathbf{R})$ satisfying 1.6(3), whose absolute value is square-integrable on $Z(\mathbf{R})\Gamma' \setminus G(\mathbf{R})$. Since $Z(\mathbf{R})\Gamma' \setminus G(\mathbf{R})$ has finite measure, this space is finite dimensional by a well-known lemma of Godement [11, Lemma 17]. This proves 1.7 for $\mathcal{A}(I, \xi, J, K)$ when $Z(\mathbf{R})\Gamma' \setminus G(\mathbf{R})$ is compact, and is the first step of the proof of 1.7 in general.

1.9. Let $a \in G(\mathbf{Q})$. Then $a \cdot I' = a' \cdot I' \cdot a^{-1}$ is an arithmetic subgroup of $G(\mathbf{Q})$, and the left translation $l_a$ by $a$ induces an isomorphism of $\mathcal{A}(I', \xi, J, K)$ onto $\mathcal{A}(a \cdot I', \xi, J, K)$. Let $\Sigma$ be a family of arithmetic subgroups of $G(\mathbf{Q})$, closed under finite intersection, whose intersection is reduced to $\{1\}$. The union $\mathcal{A}(\Sigma, \xi, J, K)$ of the spaces $\mathcal{A}(I', \xi, J, K)$ ($I' \in \Sigma$) may be identified to the inductive limit of those spaces:

$$\mathcal{A}(\Sigma, \xi, J, K) = \text{ind}_{I' \in \Sigma} \mathcal{A}(I', \xi, J, K),$$

where the inductive limit is taken with respect to the inclusions

$$j_{p,p'}: \mathcal{A}(I'', \xi, J, K) \to \mathcal{A}(I''', \xi, J, K) \quad (I''', \xi, J, K)$$

associated to the projections $I'' \setminus G(\mathbf{R}) \to I' \setminus G(\mathbf{R})$.

Assume $\Sigma$ to be stable under conjugation by $G(\mathbf{Q})$. Then $G(\mathbf{Q})$ operates on $\mathcal{A}(\Sigma, \xi, J, K)$ by left translations. Let us topologize $G(\mathbf{Q})$ by taking the elements of $\Sigma$ as a basis of open neighborhoods of $1$. Then this representation is admissible (every element is fixed under an open subgroup, and the fixed point set of every open subgroup is finite dimensional). By continuity, it extends to a continuous admissible representation of the completion $G(\mathbf{Q})_\mathbf{A}$ of $G(\mathbf{Q})$ for the topology just defined. For suitable $\Sigma$, the passage to $\mathcal{A}(\Sigma, \xi, J, K)$ amounts essentially to considering all adelic automorphic forms whose type at infinity is prescribed by $\xi, J, K$; the group $G(\mathbf{Q})_\mathbf{A}$ may be identified to the closure of $G(\mathbf{Q})$ in $G(\mathbf{A})$ and its action comes from one of $G(\mathbf{A})$). See 4.7.

1.10. Finally, we may let $\xi$ and $J$ vary and consider the space $\mathcal{A}(\Sigma, J, K)$ spanned by the $\mathcal{A}(\Sigma, \xi, J, K)$ and the space $\mathcal{A}(\Sigma, K)$ spanned by the $\mathcal{A}(\Sigma, J, K)$. They are $G(\mathbf{Q})_\mathbf{A}$-modules and $(\mathfrak{g}, K)$-modules, and these actions commute. Again, this has a natural adelic interpretation (4.8).

1.11. Hecke operators. Let $\mathcal{H}(G(\mathbf{Q}), \Gamma)$ be the Hecke algebra, over $\mathcal{C}$, of $G(\mathbf{Q})$