if we use \((a, b, c, d)\) for \((a_1, a_2, a_3, a_4)\) and \((i, j, k)\) for \((k_{12}, k_{13}, k_{23})\):

\[
\sum_{i, j, k} (-1)^{i+j+k} \binom{a+b}{b+i} \binom{a+c}{c+j} \binom{b+c}{c+k} \binom{a+d}{d-i-j} \binom{b+d}{d+i-k} \binom{c+d}{d+j+k} = \frac{(a+b+c+d)!}{a!b!c!d!}, \quad \text{integers } a, b, c, d \geq 0.
\]

The left side of (5.31) is the coefficient of \(z_1^0 z_2^0 \ldots z_n^0\) after the product of \(n(n-1)\) fractions

\[
\prod_{1 \leq i \leq n \atop i \neq j} \left(1 - \frac{z_i}{z_j}\right)^{a_i}
\]

has been fully expanded into positive and negative powers of the \(z_i\)'s. The right side of (5.31) was conjectured by Freeman Dyson in 1962 and proved by several people shortly thereafter. Exercise 86 gives a “simple” proof of (5.31).

Another noteworthy identity involving lots of binomial coefficients is

\[
\sum_{i, j, k} (-1)^{i+j+k} \binom{j+k}{j} \binom{r}{j} \binom{n}{k} \binom{m+n-j-k}{m-j} = \binom{n+r}{n} \binom{m-r}{m-n}, \quad \text{integers } m, n \geq 0.
\] (5.32)

This one, proved in exercise 83, even has a chance of arising in practical applications. But we're getting far afield from our theme of “basic identities,” so we had better stop and take stock of what we've learned.

We've seen that binomial coefficients satisfy an almost bewildering variety of identities. Some of these, fortunately, are easily remembered, and we can use the memorable ones to derive most of the others in a few steps. Table 174 collects ten of the most useful formulas, all in one place; these are the best identities to know.

## 5.2 BASIC PRACTICE

In the previous section we derived a bunch of identities by manipulating sums and plugging in other identities. It wasn’t too tough to find those derivations— we knew what we were trying to prove, so we could formulate a general plan and fill in the details without much trouble. Usually, however, out in the real world, we’re not faced with an identity to prove; we’re faced with a sum to simplify. And we don’t know what a simplified form might look like (or even if one exists). By tackling many such sums in this section and the next, we will hone our binomial coefficient tools.