To start, let’s try our hand at a few sums involving a single binomial coefficient.

**Problem 1: A sum of ratios.**

We’d like to have a closed form for

\[
\sum_{k=0}^{m} \binom{m}{k} / \binom{n}{k}, \quad \text{integers } n \geq m \geq 0.
\]

At first glance this sum evokes panic, because we haven’t seen any identities that deal with a quotient of binomial coefficients. (Furthermore, the sum involves two binomial coefficients, which seems to contradict the sentence preceding this problem.) However, just as we can use the factorial representations to reexpress a product of binomial coefficients as another product—that’s how we got identity (5.21)—we can do likewise with a quotient. In fact, we can avoid the grubby factorial representations by letting \( r = n \) and dividing both sides of equation (5.21) by \( \binom{k}{r} \binom{n}{m} \); this yields

\[
\binom{m}{k} / \binom{n}{k} = \binom{n-k}{m-k} / \binom{n}{m}.
\]

So we replace the quotient on the left, which appears in our sum, by the one on the right; the sum becomes

\[
\sum_{k=0}^{m} \binom{n-k}{m-k} / \binom{n}{m}.
\]

We still have a quotient, but the binomial coefficient in the denominator doesn’t involve the index of summation \( k \), so we can remove it from the sum. We’ll restore it later.

We can also simplify the boundary conditions by summing over all \( k \geq 0 \); the terms for \( k > m \) are zero. The sum that’s left isn’t so intimidating:

\[
\sum_{k=0}^{m} \binom{n-k}{m-k}.
\]

It’s similar to the one in identity (5.9), because the index \( k \) appears twice with the same sign. But here it’s \(-k\) and in (5.9) it’s not. The next step should therefore be obvious; there’s only one reasonable thing to do:

\[
\sum_{k \geq 0} \binom{n-k}{m-k} = \sum_{k \geq 0} \binom{n-(m-k)}{m-(m-k)} = \sum_{k \leq m} \binom{n-m+k}{k}.
\]