mod \mathcal{I}$. It is the space of complex valued functions on $G(\mathcal{Q})$ which are bi-invariant under $\mathcal{I}$ and have support in a finite union of double cosets mod $\mathcal{I}$. The product may be defined directly in terms of double cosets (see, e.g., [17]) or of convolution (see below). This algebra operates on $\mathcal{A}(\mathcal{I}, \xi, J, K)$. The effect of $\mathcal{I} a \mathcal{I}$ ($a \in G(\mathcal{Q})$) is given by $f \mapsto \sum_{b \in (\mathcal{I} a \mathcal{I})} b \cdot f$. More generally, let $\mathcal{H}(G(\mathcal{Q}), \Sigma)$ be the Hecke algebra spanned by the characteristic functions of the double cosets $\mathcal{I} a \mathcal{I}$ ($\mathcal{I}, \mathcal{I} \in \Sigma, a \in G(\mathcal{Q}))$ [17, Chapter 3]. It may be identified with the Hecke algebra $\mathcal{H}(G(\mathcal{Q}_{\mathcal{I}})$ of locally constant compactly supported functions on $G(\mathcal{Q})_{\mathcal{I}}$. This identification carries $\mathcal{H}(G(\mathcal{Q}), \mathcal{I})$ onto $\mathcal{H}(G(\mathcal{Q}_{\mathcal{I}}), \mathcal{I})$, where $\mathcal{I}$ is the closure of $\mathcal{I}$ in $G(\mathcal{Q})_{\mathcal{I}}$ [12]. The product here is ordinary convolution (which amounts to finite sums in this case). Since $\mathcal{A}(\Sigma, \xi, J, K)$ is an admissible module for $G(\mathcal{Q}_{\mathcal{I}})$, the action of $G(\mathcal{Q}_{\mathcal{I}})$ extends in the standard way to one of $\mathcal{H}(G(\mathcal{Q}_{\mathcal{I}})$). The space $\mathcal{A}(\mathcal{I}, \xi, J, K)$ is the fixed point set of $\mathcal{I}$, and the previous operation of $\mathcal{H}(G(\mathcal{Q}), \mathcal{I})$ on this space may be viewed as that of $\mathcal{H}(G(\mathcal{Q}_{\mathcal{I}}), \mathcal{I})$. For an adelic interpretation, see 4.8.

2. Automorphic forms and representations of $G(R)$. The notion of automorphic form has a simple interpretation in terms of representations (which in fact suggested its present form). To give it, we need the following known lemma (cf. [18] for the terminology).

2.1. Lemma. Let $(\pi, V)$ be a differentiable representation of $G(R)$. Let $v \in V$ be $K$-finite and $Z(\mathfrak{g})$-finite. Then the smallest $(\mathfrak{g}, K)$-submodule of $V$ containing $v$ is admissible.

Indeed, $\mathcal{H} \cdot v$ is a finite sum of spaces $\mathcal{H}^\circ \cdot w$, where $\mathcal{H}^\circ$ is the Hecke algebra of the identity component $G(\mathcal{R})^\circ$ of $G(\mathcal{R})$ and $K^\circ = K \cap G(\mathcal{R})^\circ$, and $w$ is $K^\circ$-finite and $Z(\mathfrak{g})$-finite. It suffices therefore to show that $\mathcal{H}^\circ \cdot v$ is an admissible $(\mathfrak{g}, K)$-module. By assumption, there exist an ideal $R$ of finite codimension of the enveloping algebra $U(t)$ of the Lie algebra $t$ of $K$ and an ideal $J$ of finite codimension of $Z(\mathfrak{g})$ which annihilate $v$ and moreover $U(t)/R$ is a semisimple $t$-module. Then $\mathcal{H}^\circ \cdot v$ may be identified with $U(\mathfrak{g})/U(\mathfrak{g}) \cdot R \cdot J$. By a theorem of Harish-Chandra (see [19, 2.2.1.1]), $U(\mathfrak{g})/U(\mathfrak{g}) \cdot R$ is $t$-semisimple and its $t$-isotypic submodules are finitely generated $Z(\mathfrak{g})$-modules. Hence their quotients by $J$ are finite dimensional.

2.2. We apply this to $C_c(\mathcal{I})(G(R))$, acted upon by $G(R)$ via right translations. Therefore, if $f$ is automorphic form, then $f \ast \mathcal{H}$ is an admissible $\mathcal{H}$- or $(\mathfrak{g}, K)$-module. This module consists of automorphic forms. In fact, 1.3(a) is clear, and 1.3(b) follows from 2.1; its elements are annihilated by the same ideal of $Z(\mathfrak{g})$ as $v$, whence (d). Finally, there exists $\alpha \in C_c^\infty(G)$ such that $f \ast \alpha = f$ so that $f \ast X$ satisfies 1.2(c) (with the same exponent as $f$) for all $X \in U(\mathfrak{g})$ [11, Lemma 14]. Thus the spaces

$$\mathcal{A}(\mathcal{I}, \xi, J, K) = \Sigma_{\mathcal{I}} \mathcal{A}(\mathcal{I}, \xi, J, K), \quad \mathcal{A}(\mathcal{I}, K) = \Sigma_{\mathcal{I}} \mathcal{A}(\mathcal{I}, J, K),$$

are $(\mathfrak{g}, K)$-modules and unions of admissible $(\mathfrak{g}, K)$-modules.

If $f$ is a cusp form, then $f \ast \mathcal{H}$ consists of cusp forms. Thus the subspace $\mathcal{A}(\mathcal{I}, K)$ of cusp forms is also an $\mathcal{H}$-module. If $\chi$ is a quasi-character of $Z$, then the space $\mathcal{A}(\mathcal{I}, K)_{\chi}$ of eigenfunctions for $Z$ with character $\chi$ is a direct sum of irreducible admissible $(\mathfrak{g}, K)$-modules, with finite multiplicities. In fact, after a twist by $|\chi|^{-1}$, we may assume $\chi$ to be unitary, and we are reduced to the Gelfand-Piatetski-Shapiro theorem ([7], see also [11, Theorem 2], [13, pp. 41–42]) once
we identify \( \mathcal{A}(I, K)_x \) to the space of \( K \)-finite and \( Z(g) \)-finite elements in the space \( \mathcal{O}(I\backslash G(R))_x \) of cuspidal functions in \( L^2(I\backslash G(R))_x \) (see 1.8 for the latter).

3. Some notation. We fix some notation and conventions for the rest of this paper.

3.1. \( F \) is a global field, \( O_F \) the ring of integers of \( F \), \( V \) or \( V_F \) (resp. \( V_{\infty} \), resp. \( V_f \)) the set of places (resp. archimedean places, resp. nonarchimedean places) of \( F \), \( F_v \) the completion of \( F \) at \( v \in V \), \( O_v \) the ring of integers of \( F_v \) if \( v \in V_f \). As usual, \( A \) or \( A_F \) (resp. \( A_f \)) is the ring of adeles (resp. finite adeles) of \( F \).

3.2. \( G \) is a connected reductive group over \( F \), \( Z \) the greatest \( F \)-split torus of the center of \( G \), \( \mathcal{H}_v \), the Hecke algebra of \( G_v = G(F_v) \) (\( v \in V \)) [4]. Thus \( \mathcal{H}_v \) is of the type considered in \$1$ if \( v \in V_{\infty} \) and is the convolution algebra of locally constant compactly supported functions on \( G(F_v) \) if \( v \in V_f \). We set

\[
\mathcal{H}_\infty = \bigotimes_{v \in V_{\infty}} \mathcal{H}_v, \quad \mathcal{H}_f = \bigotimes_{v \in V_f} \mathcal{H}_v, \quad \mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}_f,
\]

where the second tensor product is the restricted tensor product with respect to a suitable family of idempotents [4]. Thus \( \mathcal{H} \) is the global Hecke algebra of \( G(A) \) [4]. If \( F \) is a function field, then \( V_{\infty} \) is empty and \( \mathcal{H} = \mathcal{H}_f \).

If \( L \) is a compact open subgroup of \( G(A_f) \), we denote by \( \xi_L \) the associated idempotent, i.e., the characteristic function of \( L \) divided by the volume of \( L \) (relative to the Haar measure underlying the definition of \( \mathcal{H}_f \)). Thus \( f * \xi_L = f \) if and only if \( f \) is right invariant under \( L \).

The right translation by \( x \in G(A) \) on \( G(A) \), or on functions on \( G(A) \), is denoted \( r_x \) or \( r(x) \).

3.3. A continuous (resp. measurable) function on \( G(A) \) is cuspidal if

\[
\int_{N(F) \backslash N(A)} f(nx) \, dn = 0,
\]

for all (resp. almost all) \( x \in G(A) \), where \( N \) is the unipotent radical of any proper parabolic \( F \)-subgroup \( P \) of \( G \). It suffices to check this condition when \( P \) runs through a set of representatives of the conjugacy classes of proper maximal parabolic \( F \)-subgroups.

4. Groups over number fields.

4.1. In this section, \( F \) is a number field. An element \( \xi \in \mathcal{H} \) is said to be simple if it is of the form

\[
\xi = \xi_{\infty} \otimes \xi_f, \quad \xi_f \in \mathcal{H}_f, \xi_{\infty} \text{ idempotent in } \mathcal{H}_\infty.
\]

We let \( G_{\infty} = \prod v \in V_{\infty} G_v \) and \( g_{\infty} \) be the Lie algebra of \( G_{\infty} \), viewed as a real Lie group. We recall that \( G_{\infty} \) may be viewed canonically as the group of real points \( H(R) \) of a connected reductive group \( H \), namely the group \( H = R_{F/G} G \) obtained from \( G \) by restriction of scalars from \( F \) to \( Q \). This identification is understood when we apply the results and definitions of §§1, 2 to \( G_{\infty} \).

The group \( G(A) \) is the direct product of \( G_{\infty} \) by \( G(A_f) \). A complex valued function on \( G(A) \) is smooth if it is continuous and, if viewed as a function of two arguments \( x \in G_{\infty}, y \in G(A_f) \), it is \( C^\infty \) in \( x \) (resp. locally constant in \( y \)) for fixed \( y \) (resp. \( x \)).

4.2. Automorphic forms. Fix a maximal compact subgroup \( K_{\infty} \) of \( G_{\infty} \). A smooth function \( f \) on \( G(A) \) is a \( K_{\infty} \)-automorphic form on \( G(A) \) if it satisfies the following conditions: