The other sum $A$ is the same, but with $m$ replaced by $m-1$. Hence we have a closed form for the given sum $S$, which can be further simplified:

$$S = mA - (m-n)B = m \binom{m}{m-n} - (m-n) \binom{m+1}{m-n+1}$$

$$= \left( \frac{m-(m-n)}{m-n+1} \right) \binom{m}{m-n}$$

And this gives us a closed form for the original sum:

$$T = S \binom{m}{n}$$

$$= \frac{n}{m-n+1} \frac{m}{m-n} \binom{m}{n}$$

Even the referee can’t simplify this.

Again we use a small case to check the answer. When $m = 4$ and $n = 2$, we have

$$T = 0 \cdot \binom{4}{2} + 1 \cdot \binom{4}{3} + 2 \cdot \binom{4}{4} = 0 + \frac{2}{6} + \frac{2}{6} = \frac{2}{3},$$

which agrees with our formula $2/(4-2+1)$.

**Problem 3: From an old exam.**

Let’s do one more sum that involves a single binomial coefficient. This one, unlike the last, originated in the halls of academia; it was a problem on a take-home test. We want the value of

$$Q_n = \sum_{k \leq 2^n} \binom{2^n - k}{k} (-1)^k,$$

integer $n \geq 0$.

This one’s harder than the others; we can’t apply any of the identities we’ve seen so far. And we’re faced with a sum of $2^{1000000}$ terms, so we can’t just add them up. The index of summation $k$ appears in both indices, upper and lower, but with opposite signs. Negating the upper index doesn’t help, either; it removes the factor of $(-1)^k$, but it introduces a $2k$ in the upper index.

When nothing obvious works, we know that it’s best to look at small cases. If we can’t spot a pattern and prove it by induction, at least we’ll have