have the form $R_k$ in the resulting expression, somewhat as we did in the perturbation method of Chapter 2:

$$R_m = \sum_{k \leq m} \binom{m-k}{k} (-1)^k$$

$$= \sum_{k \leq m} \binom{m-1-k}{k} (-1)^k + \sum_{k \leq m} \binom{m-1-k}{k-1} (-1)^k$$

$$= \sum_{k \leq m} \binom{m-1-k}{k} (-1)^k + \sum_{n \geq 1, n \leq m} \binom{m-2-k}{k} (-1)^k - \binom{m-1}{k} (-1)^{m-k}$$

$$= R_{m-1} + (-1)^{2m} - R_{m-2} = (-1)^{2(m-1)} = R_{m-1} = R_{m-2}.$$  

(In the next-to-last step we’ve used the formula $\binom{-1}{n} = (-1)^n$, which we know is true when $m \geq 0$.) This derivation is valid for $m \geq 2$.

From this recurrence we can generate values of $R_m$ quickly, and we soon perceive that the sequence is periodic. Indeed,

$$R_m = \begin{cases} 1 & \text{if } m \mod 6 = 0 \\ 1 & \text{if } m \mod 6 = 1 \\ -1 & \text{if } m \mod 6 = 2 \\ 0 & \text{if } m \mod 6 = 3 \\ 1 & \text{if } m \mod 6 = 4 \\ 0 & \text{if } m \mod 6 = 5 \end{cases}$$

The proof by induction is by inspection. Or, if we must give a more academic proof, we can unfold the recurrence one step to obtain

$$R_m = (R_{m-2} - R_{m-3}) = R_{m-2} = -R_{m-3},$$

whenever $m \geq 3$. Hence $R_m = R_{m-6}$ whenever $m \geq 6$.

Finally, since $Q_n = R_{2n}$, we can determine $Q_n$ by determining $2^n \mod 6$ and using the closed form for $R_m$. When $n = 0$ we have $2^0 \mod 6 = 1$; after that we keep multiplying by 2 (mod 6), so the pattern 2, 4 repeats. Thus

$$Q_n = R_{2n} = \begin{cases} R_1 = 1, & \text{if } n = 0; \\ R_2 = 0, & \text{if } n \text{ is odd}; \\ R_4 = -1, & \text{if } n > 0 \text{ is even.} \end{cases}$$

This closed form for $Q_n$ agrees with the first four values we calculated when we started on the problem. We conclude that $Q_{1000000} = R_5 = -1$. 

Anyway those of us who’ve done warmup exercise 4 know it.