**Problem 5: A sum with three factors.**

Here’s another sum that isn’t too bad. We wish to simplify

\[ \sum_k \binom{n}{k} \binom{s}{k} k, \quad \text{integer } n \geq 0. \]

The index of summation \( k \) appears in both lower indices and with the same sign; therefore identity (5.23) in Table 169 looks close to what we need. With a bit of manipulation, we should be able to use it.

The biggest difference between (5.23) and what we have is the extra \( k \) in our sum. But we can absorb \( k \) into one of the binomial coefficients by using one of the absorption identities:

\[ \sum_k \binom{n}{k} \binom{s}{k} k = \sum_k \binom{n}{k} \binom{s-1}{k-1} s \]

\[ = s \sum_k \binom{n}{k} \binom{s-1}{k-1}. \]

We don’t care that the \( s \) appears when the \( k \) disappears, because it’s constant. And now we’re ready to apply the identity and get the closed form,

\[ s \sum_k \binom{n}{k} \binom{s-1}{k-1} = s \binom{n+s-1}{n-1}. \]

If we had chosen in the first step to absorb \( k \) into \( \binom{n}{k} \), not \( \binom{s}{k} \), we wouldn’t have been allowed to apply (5.23) directly, because \( n-1 \) might be negative; the identity requires a nonnegative value in at least one of the upper indices.

**Problem 6: A sum with menacing characteristics.**

The next sum is more challenging. We seek a closed form for

\[ \sum_{k \geq 0} \binom{n+k}{2k} \binom{2k}{k} \frac{(-1)^k}{k+1}, \quad \text{integer } n \geq 0. \]

One useful measure of a sum’s difficulty is the number of times the index of summation appears. By this measure we’re in deep trouble-\( k \) appears six times. Furthermore, the key step that worked in the previous problem—to absorb something outside the binomial coefficients into one of them—won’t work here. If we absorb the \( k+1 \) we just get another occurrence of \( k \) in its place. And not only that: Our index \( k \) is twice shackled with the coefficient \( 2 \) inside a binomial coefficient. Multiplicative constants are usually harder to remove than additive constants.