We’re lucky this time, though. The $2k$’s are right where we need them for identity (5.21) to apply, so we get

$$
\sum_{k \geq 0} \binom{n + k}{2k} \frac{(-1)^k}{k + 1} = \sum_{k \geq 0} \binom{n + k}{k} \frac{(-1)^k}{k + 1}.
$$

The two $2$’s disappear, and so does one occurrence of $k$. So that’s one down and five to go.

The $k + 1$ in the denominator is the most troublesome characteristic left, and now we can absorb it into $\binom{n}{k}$ using identity (5.6):

$$
\sum_{k \geq 0} \binom{n + k}{k} \frac{(-1)^k}{k + 1} = \sum_{k \geq 0} \binom{n + k}{k} \frac{(-1)^k}{n + 1} = \frac{1}{n + 1} \sum_{k \geq 0} \binom{n + k}{k} \frac{(-1)^k}{k + 1}.
$$

(Recall that $n \geq 0$.) Two down, four to go.

To eliminate another $k$ we have two promising options. We could use symmetry on $\binom{n+k}{k}$; or we could negate the upper index $n + k$, thereby eliminating that $k$ as well as the factor $(-1)^k$. Let’s explore both possibilities, starting with the symmetry option:

$$
\frac{1}{n + 1} \sum_{k} \binom{n + k}{k} \frac{(-1)^k}{k + 1} = \frac{1}{n + 1} \sum_{k} \binom{n + k}{n} \frac{(-1)^k}{k + 1}.
$$

Third down, three to go, and we’re in position to make a big gain by plugging into (5.24): Replacing $(1, m, n, s)$ by $(n + 1, 1, n, n)$, we get

$$
\frac{1}{n + 1} \sum_{k} \binom{n + k}{n} \frac{(-1)^k}{k + 1} = \frac{1}{n + 1} (-1)^n \binom{n - 1}{-1} = 0.
$$

Zero, eh? After all that work? Let’s check it when $n = 2$: \((\binom{2}{0}) \binom{1}{1} \binom{2}{1} \binom{1}{2} \binom{1}{3} = 1 - \frac{2}{3} + \frac{2}{3} = 0\). It checks.

Just for the heck of it, let’s explore our other option, negating the upper index of $\binom{n+k}{k}$:

$$
\frac{1}{n + 1} \sum_{k} \binom{n + k}{k} \frac{(-1)^k}{k + 1} = \frac{1}{n + 1} \sum_{k} \binom{(-n - 1)}{k} \frac{(-1)^k}{k + 1}.
$$

Now (5.23) applies, with $(1, m, n, s) \leftrightarrow (n + 1, 1, 0, -n = 1)$, and

$$
\frac{1}{n + 1} \sum_{k} \binom{-n - 1}{k} \frac{(-1)^k}{k + 1} = \frac{1}{n + 1} \binom{0}{n}.
$$