We’re lucky this time, though. The 2k’s are right where we need them for identity (5.21) to apply, so we get
\[
\sum_{k \geq 0} \binom{n+k}{2k} \frac{(-1)^k}{k+1} = \sum_{k \geq 0} \binom{n+k}{k} \binom{n}{k} \frac{(-1)^k}{k+1}.
\]
The two 2’s disappear, and so does one occurrence of k. So that’s one down and five to go.

The k+1 in the denominator is the most troublesome characteristic left, and now we can absorb it into \(\binom{n}{k}\) using identity (5.6):
\[
\sum_{k \geq 0} \binom{n+k}{k} \frac{(-1)^k}{k+1} = \sum_{k} \binom{n+k}{k} \frac{(-1)^k}{k+1} \\
= \frac{1}{n+1} \sum_{k} \binom{n+k}{k} \binom{n+1}{k+1} (-1)^k.
\]
(Recall that \(n \geq 0\).) Two down, four to go.

To eliminate another k we have two promising options. We could use symmetry on \(\binom{n+k}{k}\); or we could negate the upper index \(n+k\), thereby eliminating that k as well as the factor \((-1)^k\). Let’s explore both possibilities, starting with the symmetry option:
\[
\frac{1}{n+1} \sum_{k} \binom{n+k}{k} \binom{n-1}{k+1} (-1)^k = \frac{1}{n+1} \sum_{k} \binom{n+k}{n} \binom{n+1}{k+1} (-1)^k
\]
Third down, three to go, and we’re in position to make a big gain by plugging into (5.24): Replacing \((1, m, n, s)\) by \((n+1, 1, n, n)\), we get
\[
\frac{1}{n+1} \sum_{k} \binom{n+k}{n} \binom{n-1}{k+1} (-1)^k = \frac{1}{n+1} (-1)^n \binom{n-1}{-1} = 0.
\]
Zero, eh? After all that work? Let’s check it when \(n = 2\):
\[
\left(\frac{2}{0}\right) \left(\frac{1}{1}\right) \left(\frac{3}{2}\right) \left(\frac{1}{1}\right) \frac{1}{3} = 1 - \frac{6}{2} + \frac{9}{3} = 0.
\]
It checks.

Just for the heck of it, let’s explore our other option, negating the upper index of \(\binom{n+k}{k}\):
\[
\frac{1}{n+1} \sum_{k} \binom{n+k}{k} \binom{-n-1}{k+1} (-1)^k = \frac{1}{n+1} \sum_{k} \binom{n+1}{k} \binom{n-1}{k+1} (-1)^k.
\]
Now (5.23) applies, with \((l,m,n,s) \leftrightarrow (n+1, 1, 0, -n = 1)\), and
\[
\frac{1}{n+1} \sum_{k} \binom{-n-1}{k} \binom{n+1}{k+1} = \frac{1}{n+1} \binom{0}{n}.
\]