Hey wait. This is zero when \( n > 0 \), but it’s 1 when \( n = 0 \). Our other path to the solution told us that the sum was zero in all cases! What gives? The sum actually does turn out to be 1 when \( n = 0 \), so the correct answer is \( \{n=0\} \). We must have made a mistake in the previous derivation.

Let’s do an instant replay on that derivation when \( n = 0 \), in order to see where the discrepancy first arises. Ah yes; we fell into the old trap mentioned earlier: We tried to apply symmetry when the upper index could be negative! We were not justified in replacing \( \binom{n+k}{k} \) by \( \binom{n}{k} \) when \( k \) ranges over all integers, because this converts zero into a non-zero value when \( k < -n \). (Sorry about that.)

The other factor in the sum, \( \binom{n+1}{k+1} \), turns out to be zero when \( k < -n \), except when \( n = 0 \) and \( k = -1 \). Hence our error didn’t show up when we checked the case \( n = 2 \). Exercise 6 explains what we should have done.

**Problem 7: A new obstacle.**

This one’s even tougher; we want a closed form for

\[
\sum_{k \geq 0} \binom{n+k}{m+2k} \left( \frac{2k}{k+1} \right) \binom{-1}{k}^k, \text{ integers } m,n > 0.
\]

If \( m \) were 0 we’d have the sum from the problem we just finished. But it’s not, and we’re left with a real mess—nothing we used in Problem 6 works here. (Especially not the crucial first step.)

However, if we could somehow get rid of the \( m \), we could use the result just derived. So our strategy is: Replace \( \binom{n+k}{m+2k} \) by a sum of terms like \( \binom{1+k}{2k} \) for some nonnegative integer \( 1; \) the summand will then look like the summand in Problem 6, and we can interchange the order of summation.

What should we substitute for \( \binom{n+k}{m+2k} \)? A painstaking examination of the identities derived earlier in this chapter turns up only one suitable candidate, namely equation (5.26) in Table 169. And one way to use it is to replace the parameters \( (1,m,n,q,k) \) by \( (n+k-1,2k,m-1,0,j) \), respectively:

\[
\sum_{k \geq 0} \binom{n+k}{m+2k} \left( \frac{2k}{k+1} \right) \binom{-1}{k}^k
\]

\[
= \sum_{k \geq 0} \sum_{0 \leq j \leq n+k-1} \binom{n+k-1-j}{2k} \binom{j}{m-1} \binom{2k}{k+1} \binom{-1}{k}^k
\]

\[
= \sum_{j \geq 0} \binom{j}{m-1} \sum_{k \geq j} \binom{n+k-1-j}{2k} \binom{2k}{k+1} \binom{-1}{k}^k
\]

In the last step we’ve changed the order of summation, manipulating the conditions below the \( \sum \)'s according to the rules of Chapter 2.