in Problem 1. Replacing \( T \) by \(-k\) gives the desired expansion,

\[
S_m = \sum_{k \geq 0} \binom{n + k}{k} \frac{(-1)^k}{k + 1} \sum_{j \geq 0} \binom{m}{j} \binom{(-k - 2)}{j}^{-1}.
\]

Now the \((k + 1)^{-1}\) can be absorbed into \(\binom{n}{k}\), as planned. In fact, it could also be absorbed into \(\binom{-k - 2}{j}^{-1}\). Double absorption suggests that even more cancellation might be possible behind the scenes. Yes-expanding everything in our new summand into factorials and going back to binomial coefficients gives a formula that we can sum on \(k\):

\[
S_m = \frac{m! \cdot n!}{(m + n + 1)!} \sum_{j \geq 0} (-1)^j \binom{m + n + 1}{n + j} \sum_{k} \binom{n + 1 + j}{k + j + 1} \binom{(-k - 1)}{k}
\]

The sum over all integers \(j\) is zero, by (5.24). Hence \(-S_m\) is the sum for \(j < 0\).

To evaluate \(-S_m\) for \(j < 0\), let’s replace \(j\) by \(-k + 1\) and sum for \(k \geq 0\):

\[
S_m = \frac{m! \cdot n!}{(m + n + 1)!} \sum_{k \geq 0} (-1)^k \binom{m + n + 1}{n - k} \binom{(-k - 1)}{n}
\]

Finally (5.25) applies, and we have our answer:

\[
S_m = (-1)^n \frac{m! \cdot n!}{(m + n + 1)!} \binom{m}{n} = (-1)^n m^n n^{n-1}.
\]

Whew; we’d better check it. When \(n = 2\) we find

\[
S_m = \frac{1}{m + 1} - \frac{6}{m + 2} + \frac{6}{m + 3} = \frac{m(m - 1)}{(m + 1)(m + 2)(m + 3)}
\]

Our derivation requires \(m\) to be an integer, but the result holds for all real \(m\), because \((m + 1)^{n+1} S_m\) is a polynomial in \(m\) of degree \(\leq n\).