(b) For each $g$ the function $m \to \psi(m, g) = \phi(mg)$ is automorphic and cuspidal. Then $V_p \subseteq \mathcal{A}_p$. Since there is no point in dragging the subscript $P$ about, we change notation, letting $\pi$ be realized on $V/U$ with $U \subseteq V \subseteq \mathcal{A}_p$. We suppose that $V$ is generated by a single function $\phi$.

**Lemma 6.** Let $A$ be the centre of $M$. We may so choose $\phi$ and $V$ that there is a character $\chi$ of $A(\mathcal{A})$ satisfying $\phi(\alpha g) = \chi(\alpha)\phi(g)$ for all $g \in G(A)$ and all $\alpha \in A(\mathcal{A})$.

Since $P(A) \backslash G(A)/K$ is finite, Lemma 3 implies that any $\phi \in \mathcal{A}_p$ is $A(\mathcal{A})$-finite. Choose $V$ and the $\phi$ generating it to be such that the dimension of the span $Y$ of $\{l(\alpha)\phi|\alpha \in A(\mathcal{A})\}$ is minimal. Here $l(\alpha)$ is left translation by $\alpha$. If this dimension is one the lemma is valid. Otherwise there is an $\alpha \in A(\mathcal{A})$ and $\alpha \in C$ such that $0 < \dim(l(\alpha) - \alpha)Y < \dim Y$.

There are two possibilities. Either $(l(\alpha) - \alpha)U = (l(\alpha) - \alpha)V$ or $(l(\alpha) - \alpha)U \neq (l(\alpha) - \alpha)V$. In the second case we may replace $\phi$ by $(l(\alpha) - \alpha)\phi$, contradicting our choice. In the first we can realize $\pi$ as a subquotient of the kernel of $(l(\alpha) - \alpha)\phi$.

What we do then is choose a lattice $B$ in $A(\mathcal{A})$ such that $BA(F) \backslash A(\mathcal{A})$ is compact. Amongst all those $\phi$ and $V$ for which $Y$ has the minimal possible dimension we choose one $\phi$ for which the subgroup of $B$, defined as $\{b \in B|l(b)\phi = \beta\phi, \beta \in \mathcal{C}\}$, has maximal rank. What we conclude from the previous paragraph is that this rank must be that of $B$. Since $\phi$ is invariant under $A(\mathcal{A})$ and $BA(F) \backslash A(\mathcal{A})$ is compact, we conclude that $Y$ must now have dimension one. The lemma follows.

Choosing such a $\phi$ and $V$ we let $\nu$ be that positive character of $M(A)$ which satisfies

$$\nu(\alpha) = |\chi(\alpha)|, \quad \alpha \in A(\mathcal{A}),$$

and introduce the Hilbert space $L^2_2 = L^2_2(M(F), M(A), \chi)$ of all measurable functions $\phi$ on $M(Q) \backslash M(A)$ satisfying:

(i) For all $m \in M(A)$ and all $\alpha \in A(\mathcal{A})$, $\phi(\alpha g) = \chi(\alpha)\phi(g)$.

(ii) $\int_{A(\mathcal{A})M(Q)/M(A)} \nu^{-2}(m)|\phi(m)|^2 \, dm < \infty$.

$L^2_2$ is a direct sum of irreducible invariant subspaces, and if $\phi \in V$ then $m \to \phi(m, g)$ lies in $L^2_2$ for all $g \in G(A)$. Choose some irreducible component $H$ of $L^2_2$ on which the projection of $\phi(\cdot, g)$ is not zero for some $g \in G(A)$.

For each $\psi \in V$ define $\psi(\cdot, g)$ to be the projection of $\phi(\cdot, g)$ on $H$. It is easily seen that, for all $m_1 \in M(A), \phi'(mm_1, g) = \phi'(m, m_1g)$. Thus we may define $\phi'(g)$ by $\phi'(g) = \phi'(1, g)$. If $V' = \{\phi' | \phi \in V\}$, then we realize $\pi$ as a quotient of $V'$. However if $\delta^2$ is the modular function for $M(A)$ on $N(A)$ and $\sigma$ the representation of $M(A)$ on $H$ then $V'$ is contained in the space of Ind $\delta^{-1} \sigma$.

To prove the converse, and thereby complete the proof of the proposition, we exploit the analytic continuation of the Eisenstein series associated to cusp forms. Suppose $\pi$ is a representation of the global Hecke algebra $\mathcal{H}$, defined with respect to some maximal compact subgroup $K$ of $G(A)$. Choose an irreducible representation $\kappa$ of $K$ which is contained in $\pi$. If $E_\kappa$ is the idempotent defined by $K$ let $\mathcal{H}_\kappa = E_\kappa \mathcal{H} E_\kappa$ and let $\pi_\kappa$ be the irreducible representation of $\mathcal{H}_\kappa$ on the $\kappa$-isotypical subspace of $\pi$. To show that $\pi$ is an automorphic representation, it is sufficient to show that $\pi_\kappa$ is a constituent of the representation of $\mathcal{H}_\kappa$ on the space of automorphic
forms of type $\kappa$. To lighten the burden on the notation, we henceforth denote $\pi_\kappa$ by $\pi$ and $\mathcal{H}_\kappa$ by $\mathcal{H}$.

Suppose $P$ and the cuspidal representation $\sigma$ of $M(A)$ are given. Let $L$ be the lattice of rational characters of $M$ defined over $F$ and let $L_\mathcal{C} = L \otimes \mathbb{C}$. Each element $\mu$ of $L_\mathcal{C}$ defines a character $\xi_\mu$ of $M(A)$. Let $I_\mu$ be the $\kappa$-isotypical subspace of Ind $\xi_\mu \sigma$ and let $I = I_0$. We want to show that if $\pi$ is a constituent of the representation on $I$ then $\pi$ is a constituent of the representation of $\mathcal{H}$ on the space of automorphic forms of type $\kappa$.

If $\{g_i\}$ is a set of coset representations for $P(A) \backslash G(A) / K$ then we may identify $I_\mu$ with $I$ by means of the map $\varphi \rightarrow \varphi_\mu$ with

$$\varphi_\mu(nmg_i, k) = \xi_\mu(m) \varphi(nmg_i, k).$$

In other words we have a trivialization of the bundle $\{I_\mu\}$ over $L_\mathcal{C}$, and we may speak of a holomorphic section or of a section vanishing at $\mu = 0$ to a certain order. These notions do not depend on the choice of the $g_i$, although the trivialization does.

There is a neighbourhood $U$ of $\mu = 0$ and a finite set of hyperplanes passing through $U$ so that for $\mu$ in the complement of these hyperplanes in $U$ the Eisenstein series $E(\varphi)$ is defined for $\varphi$ in $I_\mu$. To make things simpler we may even multiply $E$ by a product of linear functions and assume that it is defined on all of $U$. Since it is only the modified function that we shall use, we may denote it by $E$, although it is no longer the true Eisenstein series. It takes values on the space of automorphic forms and thus $E(\varphi)$ is a function $g \rightarrow E(g, \varphi)$ on $G(A)$. It satisfies

$$E(\rho_\mu(h)\varphi) = r(h)E(\varphi)$$

if $h \in \mathcal{H}$ and $\rho_\mu$ is Ind $\xi_\mu \sigma$. In addition, if $\varphi_\mu$ is a holomorphic section of $\{I_\mu\}$ in a neighbourhood of $0$ then $E(g, \varphi_\mu)$ is holomorphic in $\mu$ for each $g$, and the derivatives of $E(\varphi_\mu)$, taken pointwise, continue to be in $\mathcal{A}$.

Let $I_\mu$ be the space of germs of degree $r$ at $\mu = 0$ of holomorphic sections of $I$. Then $\varphi_\mu \rightarrow \rho_\mu(h)\varphi_\mu$ defines an action of $\mathcal{H}$ on $I_\mu$. If $s \leq r$ the natural map $I_s \rightarrow I_\mu$ is an $\mathcal{H}$-homomorphism. Denote its kernel by $I_s^\mu$. Certainly $I_0 = I$. Choosing a basis for the linear forms on $L_\mathcal{C}$ we may consider power series with values in the $\kappa$-isotypical subspace of $\mathcal{A}$, $\sum_{|\alpha| \leq r} \mu^\alpha \phi_\alpha$. $\mathcal{H}$ acts by right translation in this space and the representation so obtained is, of course, a direct sum of that on the $\kappa$-isotypical automorphic forms. Moreover $\varphi_\mu \rightarrow E(\varphi_\mu)$ defines an $\mathcal{H}$-homomorphism $\lambda$ from $I_\mu$ to this space. To complete the proof of the proposition all one needs is the Jordan-Hölder theorem and the following lemma.

**Lemma 7.** For $r$ sufficiently large the kernel of $\lambda$ is contained in $I_0^\mu$.

Since we are dealing with Eisenstein series associated to a fixed $P$ and $\sigma$ we may replace $E$ by the sum of its constant terms for the parabolic associated to $P$, modifying $\lambda$ accordingly. All of these constant terms vanish identically if and only if $E$ itself does. If $Q_1, \ldots, Q_m$ is a set of representatives for the classes of parabolics associated to $P$ let $E_i(\varphi)$ be the constant term along $Q_i$. We may suppose that $M$ is a Levi factor of each $Q_i$. Define $\nu(m)$ for $m \in M(A)$ by $\xi_\mu(m) = e^{\langle \mu, \nu(m) \rangle}$. Thus $\nu(m)$ lies in the dual of $L_\mathcal{R}$. If $\varphi \in I_\mu$, the function $E_\lambda(\varphi)$ has the form