5.3 TRICKS OF THE TRADE

Let’s look next at three techniques that significantly amplify the methods we have already learned.

**Trick 1: Going halves.**

Many of our identities involve an arbitrary real number \( r \). When \( r \) has the special form “integer minus one half,” the binomial coefficient \( \binom{r}{k} \) can be written as a quite different-looking product of binomial coefficients. This leads to a new family of identities that can be manipulated with surprising ease.

One way to see how this works is to begin with the duplication formula

\[
\frac{r^k}{k!} \left( r - \frac{1}{2} \right)^k = \frac{(2r)^{2k} / 2^{2k}}{2 \cdot 2 \cdots 2}, \quad \text{integer } k \geq 0. \tag{5.34}
\]

This identity is obvious if we expand the falling powers and interleave the factors on the left side:

\[
r(r - \frac{1}{2})(r - 1)(r - \frac{3}{2}) \cdots (r - k + 1)(r - k + \frac{1}{2}) = \frac{(2r)(2r - 1) \cdots (2r - 2k + 1)}{2 \cdot 2 \cdots 2}
\]

Now we can divide both sides by \( k! \), and we get

\[
\binom{r}{k} \binom{r - 1/2}{k} = \frac{(2r)(2k)}{k^2} \frac{2^{2k}}{2 \cdot 2 \cdots 2}, \quad \text{integer } k. \tag{5.35}
\]

If we set \( k = r = n \), where \( n \) is an integer, this yields

\[
\binom{n - 1/2}{n} = \binom{2n}{n} / 2^{2n}, \quad \text{integer } n. \tag{5.36}
\]

And negating the upper index gives yet another useful formula,

\[
\binom{-1/2}{n} = \binom{-1}{4} \binom{2n}{n}, \quad \text{integer } n. \tag{5.37}
\]

For example, when \( n = 4 \) we have

\[
\binom{-1/2}{4} = \frac{(-1/2)(-3/2)(-5/2)(-7/2)}{4!} = \frac{\binom{-1}{4} \cdot 1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 3 \cdot 5 \cdot 7} = \binom{-1}{4} \binom{8}{4}.
\]

Notice how we’ve changed a product of odd numbers into a factorial.