Identity (5.35) has an amusing corollary. Let \( r = \frac{1}{2}n \), and take the sum over all integers \( k \). The result is

\[
\sum_k \binom{n}{2k} (2k) 2^{-2k} = \sum_k \binom{n/2}{k} \binom{(n-1)/2}{k}\n\]

\[
= \binom{n-1/2}{\lfloor n/2 \rfloor}, \quad \text{integer } n \geq 0 \tag{5.38}
\]

by (5.23), because either \( n/2 \) or \( (n-1)/2 \) is \( \lfloor n/2 \rfloor \), a nonnegative integer!

We can also use Vandermonde's convolution (5.27) to deduce that

\[
\sum_k \binom{-1/2}{k} \binom{-1/2}{n-k} = \binom{-1}{n} = (-1)^n, \quad \text{integer } n \geq 0.
\]

Plugging in the values from (5.37) gives

\[
\binom{-1/2}{k} \binom{-1/2}{n-k} = \binom{-1}{4} \binom{2k}{k} \binom{-1}{n-k} \binom{2(n-k)}{n-k}
\]

\[
= \frac{(-1)^n}{4^n} \binom{2k}{k} \binom{2n-2k}{n-k};
\]

this is what sums to \((-1)^n\). Hence we have a remarkable property of the "middle" elements of Pascal's triangle:

\[
\sum_k \binom{2k}{k} \binom{2n-2k}{n-k} = 4^n, \quad \text{integer } n \geq 0. \tag{5.39}
\]

For example, \( \binom{2}{2} + \binom{2}{1} \binom{2}{1} + \binom{2}{0} \binom{2}{2} = 1 \cdot 2 \cdot 2 + 2 \cdot 1 = 64 = 4^3 \).

These illustrations of our first trick indicate that it's wise to try changing binomial coefficients of the form \( \binom{n}{k} \) into binomial coefficients of the form \( \binom{n-1/2}{k} \), where \( n \) is some appropriate integer (usually 0, 1, or \( k \)); the resulting formula might be much simpler.

**Trick 2: High-order differences.**

We saw earlier that it's possible to evaluate partial sums of the series \( \binom{n}{k} (-1)^k \), but not of the series \( \binom{k}{n} \). It turns out that there are many important applications of binomial coefficients with alternating signs, \( \binom{n}{k} (-1)^k \). One of the reasons for this is that such coefficients are intimately associated with the difference operator \( \Delta \) defined in Section 2.6.

The difference \( \Delta f \) of a function \( f \) at the point \( x \) is

\[
\Delta f(x) = f(x+1) - f(x);
\]