In each of Problems 5 through 12 find the general solution of the given differential equation. In Problems 11 and 12 $g$ is an arbitrary continuous function.

5. $y'' + y = \tan t$, \quad $0 < t < \pi/2$
6. $y'' + 9y = 9 \sec^2 t$, \quad $0 < t < \pi/6$
7. $y'' + 4y' + 4y = t^2 e^{-2t}$, \quad $t > 0$
8. $y'' + 4y = 3 \csc 2t$, \quad $0 < t < \pi/2$
9. $4y'' + y = 2 \sec(t/2)$, \quad $-\pi < t < \pi$
10. $y'' - 2y' + y = e^t/(1 + t^2)$
11. $y'' - 5y' + 6y = g(t)$
12. $y'' + 4y = g(t)$

In each of Problems 13 through 20 verify that the given functions $y_1$ and $y_2$ satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20 $g$ is an arbitrary continuous function.

13. $t^2y'' - 2y = 3t^2 - 1$, \quad $t > 0$; \quad $y_1(t) = t^2$, \quad $y_2(t) = t^{-1}$
14. $t^2y'' - t(t + 2)y' + (t + 2)y = 2t^3$, \quad $t > 0$; \quad $y_1(t) = t$, \quad $y_2(t) = te^t$
15. $ty'' - (1 + t)y' + y = t^2 e^{t^2}$, \quad $t > 0$; \quad $y_1(t) = 1 + t$, \quad $y_2(t) = e^t$
16. $(1 - t)y'' + ty' - y = 2(t - 1) e^{-t}$, \quad $0 < t < 1$; \quad $y_1(t) = e^t$, \quad $y_2(t) = t$
17. $x^2y'' - 3xy' + 4y = x^2 \ln x$, \quad $x > 0$; \quad $y_1(x) = x^2$, \quad $y_2(x) = x^2 \ln x$
18. $x^2y'' + xy' + (x^2 - 0.25)y = 3x^{1/2} \sin x$, \quad $x > 0$; \quad $y_1(x) = x^{-1/2} \sin x$, \quad $y_2(x) = x^{-1/2} \cos x$
19. $(1 - x)y'' + xy' - y = g(x)$, \quad $0 < x < 1$; \quad $y_1(x) = e^x$, \quad $y_2(x) = x$
20. $x^2y'' + xy' + (x^2 - 0.25)y = g(x)$, \quad $x > 0$; \quad $y_1(x) = x^{-1/2} \sin x$, \quad $y_2(x) = x^{-1/2} \cos x$

21. Show that the solution of the initial value problem

\[ L[y] = y'' + p(t)y' + g(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0 \]  

(i)

can be written as $y = u(t) + v(t)$, where $u$ and $v$ are solutions of the two initial value problems

\[ L[u] = 0, \quad u(t_0) = y_0, \quad u'(t_0) = y'_0, \]  

(ii)

\[ L[v] = g(t), \quad v(t_0) = 0, \quad v'(t_0) = 0, \]  

(iii)

respectively. In other words, the nonhomogeneities in the differential equation and in the initial conditions can be dealt with separately. Observe that $u$ is easy to find if a fundamental set of solutions of $L[u] = 0$ is known.

22. By choosing the lower limit of integration in Eq. (28) in the text as the initial point $t_0$, show that $Y(t)$ becomes

\[ Y(t) = \int_0^t \frac{y_1(s)y_2(t) - y_1(t)y_2(s)}{y_1(s)y_2(t) - y_1(t)y_2(s)} g(s) \, ds. \]

Show that $Y(t)$ is a solution of the initial value problem

\[ L[y] = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0. \]

Thus $Y$ can be identified with $v$ in Problem 21.

23. (a) Use the result of Problem 22 to show that the solution of the initial value problem

\[ y'' + y = g(t), \quad y(t_0) = 0, \quad y'(t_0) = 0 \]  

(i)

is

\[ y = \int_0^t \sin(t - s)g(s) \, ds. \]  

(ii)

(b) Find the solution of the initial value problem

\[ y'' + y = g(t), \quad y(0) = y_0, \quad y'(0) = y'_0. \]