if we apply A again, we get the second difference

$$\Delta^2 f(x) = Af(x + 1) = f(x+2) - f(x+1) - (f(x+1) - f(x)) = f(x+2) - 2f(x+1) + f(x),$$

which is analogous to the second derivative. Similarly, we have

$$\Delta^3 f(x) = f(x+3) - 3f(x+2) + 3f(x+1) - f(x);$$

$$\Delta^4 f(x) = f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x);$$

and so on. Binomial coefficients enter these formulas with alternating signs.

In general, the nth difference is

$$\Delta^n f(x) = \sum_k \binom{n}{k} (-1)^{n-k} f(x+k), \quad \text{integer } n \geq 0. \tag{5.40}$$

This formula is easily proved by induction, but there's also a nice way to prove it directly using the elementary theory of operators. Recall that Section 2.6 defines the shift operator E by the rule

$$Ef(x) = f(x+1);$$

hence the operator A is $E - 1$, where 1 is the identity operator defined by the rule $1 f(x) = f(x)$. By the binomial theorem,

$$A^n = (E - 1)^n = \sum_k \binom{n}{k} E^k (-1)^{n-k}.$$ 

This is an equation whose elements are operators; it is equivalent to (5.40), since $E^k$ is the operator that takes $f(x)$ into $f(x + k)$.

An interesting and important case arises when we consider negative falling powers. Let $f(x) = (x - 1)^{-1} = 1/x$. Then, by rule (2.45), we have $Af(x) = (-1)(x-1)^{-2}$, $\Delta^2 f(x) = (x)(-2)(x-1)^{-3}$, and in general

$$\Delta^n ((x-1)^{-1}) = (-1)^n (x-1)^{-n-1} = (-1)^n \frac{n!}{x(x+1) \ldots (x+n)}.$$

Equation (5.40) now tells us that

$$\sum_k \binom{n}{k} \frac{(-1)^k}{x+k} = \frac{n!}{x(x+1) \ldots (x+n)}$$

$$= x^{-1} \left( \frac{x+n}{n} \right)^{-1}, \quad x \notin \{0, -1, \ldots, -n\}. \tag{5.41}$$