(ii) \( W_{\psi(h)\phi}(g) = W_{\psi}(gh) \), for all \( g, h \in G \).

We have the following important result due to Gelfand-Kazhdan for the case \( k \) nonarchimedean, and Shalika for general local fields ([1], [2]).

**Uniqueness Theorem.** For each irreducible admissible smooth representation \((\pi, V)\), there exists at most one \( W(\pi, \phi) \) (for fixed \( \phi \)).

For \( k \) archimedean we assume that \((\pi, V)\) is a unitarizable representation and \( V = \{ x \in H \mid (\mathcal{D}x, \mathcal{D}x) < \infty \forall \mathcal{D} \in \text{enveloping algebra} \} \). Here \( H \) means the completion of \( V \) with respect to the inner product \((x, x)\). We assume also that \( W_{\phi}(1) \) is a continuous linear functional on \( V \) with respect to the topology defined by semi-norms \((\mathcal{D}x, \mathcal{D}x), \mathcal{D} \in \text{enveloping algebra} \).

Returning to the global case, we point out that the preceding discussion easily implies uniqueness of global Whittaker models (defined in the obvious way).

**2. Global Fourier analysis.** Let \((\pi, V)\) be admissible irreducible cuspidal as before, \( \phi \in V \). Then we can define

\[
W_{\phi}(g) = \int_{X_{A} \times X_{A}} \phi(xg)\phi^{-1}(x) \, dx.
\]

Global Fourier analysis says that this “Fourier transform” defines a cusp-form uniquely. In the classical setting this is due to Hecke; for \( n = 2 \) it is proved in Jacquet-Langlands [3]; for \( n > 2 \) it is due independently to Piatetski-Shapiro [4] and Shalika [2]. The proof is motivated by a corresponding result over a finite field due to S. I. Gelfand [5].

It is now easy to see that these results imply Theorem (1), since

\[
\dim \text{Hom}_{c}(V, W(\pi, \phi)) = 1 \geq \dim \text{Hom}_{c}(V, L_{0}).
\]

We now turn to the proof of the strong multiplicity one theorem. First we discuss the case \( n = 2 \); we need the following

**Small Lemma.** (1) Assume \( k \) local, \((\pi_{1}, V_{1}), (\pi_{2}, V_{2})\) two irreducible admissible representations with Whittaker models. Then there exist \( v_{1} \in V_{1}, v_{2} \in V_{2} \) such that

\[
W_{v_{1}}(x \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = W_{v_{2}}(x \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) \quad (W_{v_{i}} \in W(\pi_{i}, \lambda)).
\]

(2) If \( k = \mathbb{R} \) or \( \mathbb{C} \) we assume that \((\pi_{1}, H_{1})\) and \((\pi_{2}, H_{2})\) are irreducible infinite-dimensional unitary representations. Denote by \( V_{1} (V_{2}) \) the set of all smooth vectors in \( H_{1} (H_{2}) \). Then there exist \( v_{1} \in V_{1}, v_{2} \in V_{2} \) such that

\[
W_{v_{1}}(x \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = W_{v_{2}}(x \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) \quad (W_{v_{i}} \in W(\pi_{i}, \phi)).
\]

**Proof.** For \( k \) a local nonarchimedean field it is known that \( V \) contains all Schwartz-Bruhat functions with compact support in \( k^{a} \). Hence we have what we want.

Now let \( k = \mathbb{R} \) or \( \mathbb{C} \). The Kirillov theorem (see [8, p. 221]) says that each irreducible infinite-dimensional unitary representation of \( \text{GL}(2, k) \) remains irreducible after restriction on the subgroup \( \{ \mathbf{GL}(2) \} = P \) and hence as a representation of \( P \) is isomorphic to the standard representation of \( P \). Hence, if \( \phi(x) \) is a \( C^{\infty} \)-function with compact support then there exist \( v_{1} \in V_{1}, v_{2} \in V_{2} \) such that
\[ W_{\nu}(x^0) = \varphi(x). \]

**Remark.** Assume that for a unitary representation with a Whittaker model the inner product can be written as an integral similar to the case for \( n = 2 \). Using this result we can prove the "small lemma" for any \( n \) as we did for \( n = 2 \). This implies the strong multiplicity one theorem for any \( n \).

Next we give the formula for recovering \( \varphi \) from its Whittaker model due to Jacquet-Langlands, for \( \text{GL}(2, A) \):

\[ (*) \quad \varphi(g) = \sum_{j \in k^*} W_j(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g). \]

Now suppose \( \pi_1, \pi_2 \) satisfy the hypotheses of the theorem. To prove the assertion, it is enough to produce a nonzero \( \varphi \in V_1 \cap V_2 \), since then the irreducibility of \( (\pi_i, V_i) \) implies equality. Further, since \( B \backslash G \) is dense in \( G \backslash G_A \), it is enough to produce two functions (nonzero) \( \varphi_i \in V_i \) which are equal on \( B_A \) (as usual \( B \) is the group of upper triangular matrices).

From the properties of Whittaker models and \( (*) \), it is enough to produce Whittaker functions \( W_1, W_2 \) such that \( W_1(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} x) = W_2(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} x) \), \( x \in A^* \). One can suppose such \( W_i \) are of the form \( \prod_{j \in k^*} \varphi_j \) and then it suffices to construct the appropriate \( W_i \) at a finite number of places (by assumption). But then one can use the small lemma. This type of argument was found independently here by Shalika and in Moscow.

For \( n \geq 3 \), we need a similar small lemma (Gelfand-Kazhdan): Suppose \( k \) local, nonarchimedean, \( (\pi_i, V_i) \), \( i = 1, 2 \), irreducible admissible representations with Whittaker models. There exist \( \psi \) such that

\[ W_{\psi}(h^0) = W_{\psi}(h^0), \quad \text{all } h \in \text{GL}(n - 1). \]

One can then employ induction using arguments similar to the case \( n = 2 \), in order to prove the general case. It should be possible to prove this lemma also for \( k \) archimedean; then the restriction we made that \( S \) contains no infinite places could be removed.

Now suppose \( G \) is quasi-split and satisfies the transitivity condition:

\[ T(A) \text{ acts transitively on } \prod_{\alpha \text{ simple root}} X_{\alpha}(A). \]

Here \( T \) is a maximal \( k \)-torus in a Borel group, \( X_{\alpha} = X_{\alpha} - \{ I \} \) where \( X_{\alpha} \) is the root group associated to the simple root \( \alpha \).

Define an automorphic cuspidal irreducible representation \( (\pi, V) \) to be hypercuspidal (degenerate cuspidal) if

\[ W_{\psi}(g) = \int_{X_{\pi} \times X_{\psi}} \varphi(xg)\psi^{-1}(x) \, dx = 0 \]

for all \( \varphi \in V \). Holomorphic cusp forms lifted from symmetric spaces which contain no copies of \( H = \{ \text{Im } z > 0 \} \) are of this type.

A cuspidal automorphic form will be called generic if it is orthogonal to all hypercuspidal automorphic forms (under the usual scalar product \( \int_{CG_{\chi}(A)} \varphi \psi \, dg \)).

Counterexamples to the Ramanujan conjecture given during this conference by Howe and the author are hypercuspidal forms [6]. The author does not wish to kill