The Newton series for \( f(x) \) is therefore

\[
f(x) = \Delta^d f[0] \frac{x^d}{d!} + \Delta^{d-1} f[0] \binom{x}{d-1} + \ldots + \Delta f[0] \binom{x}{1} + f[0] \binom{x}{0}
\]

For example, suppose \( f(x) = x^3 \). It’s easy to calculate

\[
f(0) = 0, \quad f(1) = 1, \quad f(2) = 8, \quad f(3) = 27;
\]
\[
\Delta f[0] = 1, \quad \Delta f[1] = 7, \quad \Delta f[2] = 19;
\]
\[
\Delta^2 f[0] = 6, \quad \Delta^2 f[1] = 12;
\]
\[
\Delta^3 f[0] = 6.
\]

So the Newton series is

\[
x^3 = 6 \binom{x}{0} + 6 \binom{x}{1} + 1 \binom{x}{2} + 0 \binom{x}{3}.
\]

Our formula \( \Delta^n f(0) = c_n \) can also be stated in the following way, using (5.40) with \( x = 0 \):

\[
\sum_k \binom{n}{k} (-1)^k \left( c_0 \binom{k}{0} + c_1 \binom{k}{1} + c_2 \binom{k}{2} + \ldots \right) = (-1)^n c_n, \quad \text{integer } n \geq 0.
\]

Here \( (c_0, c_1, c_2, \ldots) \) is an arbitrary sequence of coefficients; the infinite sum \( c_0 \binom{k}{0} + c_1 \binom{k}{1} + c_2 \binom{k}{2} + \ldots \) is actually finite for all \( k \geq 0 \), so convergence is not an issue. In particular, we can prove the important identity

\[
\sum_k \binom{n}{k} (-1)^k (a_0 + a_1 k + \ldots + a_n k^n) = (-1)^n n! a_n, \quad \text{integer } n \geq 0.
\]

(5.42)

because the polynomial \( a_0 + a_1 k + \ldots + a_n k^n \) can always be written as a Newton series \( c_0 \binom{k}{0} + c_1 \binom{k}{1} + \ldots + c_n \binom{k}{n} \) with \( c_n = n! a_n \).

Many sums that appear to be hopeless at first glance can actually be summed almost trivially by using the idea of nth differences. For example, let’s consider the identity

\[
\sum_k \binom{n}{k} \binom{r - sk}{n} = s^n, \quad \text{integer } n \geq 0.
\]

(5.43)

This looks very impressive, because it’s quite different from anything we’ve seen so far. But it really is easy to understand, once we notice the telltale factor \( \binom{r}{n} (-1)^k \) in the summand, because the function

\[
f[k] = \binom{r - sk}{n} = \frac{1}{n!} (-1)^n s^n k^n + \ldots = (-1)^n s^n \binom{k}{n} + \ldots
\]