as Hooke's law. Thus we write \( F_s = -kL \), where the constant of proportionality \( k \) is called the spring constant, and the minus sign is due to the fact that the spring force acts in the upward (negative) direction. Since the mass is in equilibrium, the two forces balance each other, which means that

\[ mg - kL = 0. \tag{2} \]

For a given weight \( w = mg \), one can measure \( L \) and then use Eq. (2) to determine \( k \). Note that \( k \) has the units of force/length.

In the corresponding dynamic problem we are interested in studying the motion of the mass when it is acted on by an external force or is initially displaced. Let \( u(t) \), measured positive downward, denote the displacement of the mass from its equilibrium position at time \( t \); see Figure 3.8.1. Then \( u(t) \) is related to the forces acting on the mass through Newton’s law of motion,

\[ mu''(t) = f(t), \tag{3} \]

where \( u'' \) is the acceleration of the mass and \( f \) is the net force acting on the mass. Observe that both \( u \) and \( f \) are functions of time. In determining \( f \) there are four separate forces that must be considered:

1. The weight \( w = mg \) of the mass always acts downward.
2. The spring force \( F_s \) is assumed to be proportional to the total elongation \( L + u \) of the spring and always acts to restore the spring to its natural position. If \( L + u > 0 \), then the spring is extended, and the spring force is directed upward. In this case

\[ F_s = -k(L + u). \tag{4} \]

On the other hand, if \( L + u < 0 \), then the spring is compressed a distance \(|L + u|\), and the spring force, which is now directed downward, is given by \( F_s = k|L + u| \). However, when \( L + u < 0 \), it follows that \(|L + u| = -(L + u)\), so \( F_s \) is again given by Eq. (4). Thus, regardless of the position of the mass, the force exerted by the spring is always expressed by Eq. (4).

3. The damping or resistive force \( F_d \) always acts in the direction opposite to the direction of motion of the mass. This force may arise from several sources: resistance from the air or other medium in which the mass moves, internal energy dissipation due to the extension or compression of the spring, friction between the mass and the guides (if any) that constrain its motion to one dimension, or a mechanical device (dashpot) that imparts a resistive force to the mass. In any case, we assume that the resistive force is proportional to the speed \(|du/dt|\) of the mass; this is usually referred to as viscous damping. If \( du/dt > 0 \), then \( u \) is increasing, so the mass is moving downward. Then \( F_d \) is directed upward and is given by

\[ F_d(t) = -\gamma u'(t), \tag{5} \]

where \( \gamma \) is a positive constant of proportionality known as the damping constant. On the other hand, if \( du/dt < 0 \), then \( u \) is decreasing, the mass is moving upward, and \( F_d \) is directed downward. In this case, \( F_d = \gamma |u'(t)| \); since \(|u'(t)| = -u'(t)\),

\footnote{Robert Hooke (1635–1703) was an English scientist with wide-ranging interests. His most important book, \textit{Micrographia}, was published in 1665 and described a variety of microscopical observations. Hooke first published his law of elastic behavior in 1676 as an anagram: \textit{ceiiinosssttuv}; in 1678 he gave the solution \textit{ut tensio sic vis}, which means, roughly, “as the force so is the displacement.”}