One example of a convergent Newton series is provided by the binomial theorem. Let \( g(x) = (1 + z)^x \), where \( z \) is a fixed complex number such that \( |z| < 1 \). Then \( \Delta g(x) = (1 + z)^{x+1} - (1 + z)^x = z(1 + z)^x \), hence \( \Delta^n g(x) = z^n (1 + z)^x \). In this case the infinite Newton series

\[
g(a + x) = \sum_n \Delta^n g(a) \binom{x}{n} = (1 + z)^a \sum_n \binom{x}{n} z^n
\]

converges to the “correct” value \( (1 + z)^{a+x} \), for all \( x \).

James Stirling tried to use Newton series to generalize the factorial function to noninteger values. First he found coefficients \( S_n \) such that

\[
x! = \sum_n S_n \binom{x}{n} = S_0 \binom{x}{0} + S_1 \binom{x}{1} + S_2 \binom{x}{2} + \cdots \tag{5.46}
\]

is an identity for \( x = 0, x = 1, x = 2, \) etc. But he discovered that the resulting series doesn’t converge except when \( x \) is a nonnegative integer. So he tried again, this time writing

\[
\ln x! = \sum_n s_n \binom{x}{n} = s_0 \binom{x}{0} + s_1 \binom{x}{1} + s_2 \binom{x}{2} + \cdots \tag{5.47}
\]

Now \( A(\ln x!) = \ln(x + 1)! - \ln x! = \ln(x + 1) \), hence

\[
s_n = \Delta^n (\ln x!) \big|_{x=0} = \Delta^{n-1} (\ln(x + 1)) \big|_{x=0} = \sum_k \binom{n-1}{k} (-1)^{n-1-k} \ln(k + 1)
\]

by (5.40). The coefficients are therefore \( s_0 = s_1 = 0; s_2 = \ln 2; s_3 = \ln 3/2 \); \( s_4 = \ln 4 - 3 \ln 3 + 3 \ln 2 = \ln 4/3 \); etc. In this way Stirling obtained a series that does converge (although he didn’t prove it); in fact, his series converges for all \( x > -1 \). He was thereby able to evaluate \( \frac{1}{2}! \) satisfactorily. Exercise 88 tells the rest of the story.

**Trick 3: Inversion.**

A special case of the rule (5.45) we’ve just derived for Newton’s series can be rewritten in the following way:

\[
g(n) = \sum_k \binom{n}{k} (-1)^k f(k) \iff f(n) = \sum_k \binom{n}{k} (-1)^k g(k) \tag{5.48}
\]

(Proofs of convergence not invented until the nineteenth century.)