This observation is easy to prove, since those table entries are the $h(n, n-2)$'s, and we have

$$h(n, n-2) = \binom{n}{n-2} 2^i = \binom{n}{2}.$$ 

It also seems that the numbers in the first two columns differ by $f(1)$. Is this always true? Yes,

$$h(n, 0) - h(n, 1) = n_i - n(n-1)_i$$

$$= \left( n! \sum_{0 \leq k \leq n} \frac{(-1)^k}{k!} \right) - \left( n(n-1)! \sum_{0 \leq k \leq n-1} \frac{(-1)^k}{k!} \right)$$

$$= n! \frac{(-1)^n}{n!} = (-1)^n$$

In other words, $n_i = n(n-1)_i + (-1)^n$. This is a much simpler recurrence for the derangement numbers than we had before.

Now let’s invert something else. If we apply inversion to the formula

$$\sum_k \binom{n}{k} \frac{(-1)^k}{x+k} = \frac{1}{x} \left( \frac{x+n}{n} \right)^{-1}$$

that we derived in (5.41), we find

$$\frac{x}{x+n} = \sum_{k \geq 0} \binom{n}{k} (-1)^k \binom{x+k}{k}^{-1}.$$ 

This is interesting, but not really new. If we negate the upper index in $(x+k)$, we have merely discovered identity (5.33) again.

### 5.4 Generating Functions

We come now to the most important idea in this whole book, the notion of a generating function. An infinite sequence $(a_0, a_1, a_2, \ldots)$ that we wish to deal with in some way can conveniently be represented as a power series in an auxiliary variable $z$,

$$A(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k \geq 0} a_k z^k.$$ 

(5.52)

It's appropriate to use the letter $z$ as the name of the auxiliary variable, because we'll often be thinking of $z$ as a complex number. The theory of complex variables conventionally uses $z$ in its formulas; power series (a.k.a. analytic functions or holomorphic functions) are central to that theory.