5.4 GENERATING FUNCTIONS

We will be seeing lots of generating functions in subsequent chapters. Indeed, Chapter 7 is entirely devoted to them. Our present goal is simply to introduce the basic concepts, and to demonstrate the relevance of generating functions to the study of binomial coefficients.

A generating function is useful because it’s a single quantity that represents an entire infinite sequence. We can often solve problems by first setting up one or more generating functions, then by fooling around with those functions until we know a lot about them, and finally by looking again at the coefficients. With a little bit of luck, we’ll know enough about the function to understand what we need to know about its coefficients.

If \( A(z) \) is any power series \( \sum_{k \geq 0} a_k z^k \), we will find it convenient to write

\[
[z^n] A(z) = a_n ;
\]  

(5.53)

in other words, \([z^n] A(z)\) denotes the coefficient of \( z^n \) in \( A(z) \).

Let \( A(z) \) be the generating function for \((a_0, a_1, a_2, \ldots)\) as in (5.52), and let \( B(z) \) be the generating function for another sequence \((b_0, b_1, b_2, \ldots)\). Then the product \( A(z) B(z) \) is the power series

\[
(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) = a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \cdots;
\]

the coefficient of \( z^n \) in this product is

\[
a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.
\]

Therefore if we wish to evaluate any sum that has the general form

\[
c_n = \sum_{k=0}^{n} a_k b_{n-k},
\]

(5.54)

and if we know the generating functions \( A(z) \) and \( B(z) \), we have

\[
c_n = [z^n] A(z) B(z)
\]

The sequence \((c_n)\) defined by (5.54) is called the convolution of the sequences \((a_n)\) and \((b_n)\); two sequences are “convolved” by forming the sums of all products whose subscripts add up to a given amount. The gist of the previous paragraph is that convolution of sequences corresponds to multiplication of their generating functions.