The flow of current in the circuit is governed by Kirchhoff’s second law: In a closed circuit, the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit.

According to the elementary laws of electricity, we know that:

The voltage drop across the resistor is $IR$.

The voltage drop across the capacitor is $Q/C$.

The voltage drop across the inductor is $LdI/dt$.

Hence, by Kirchhoff’s law,

$$L \frac{dI}{dt} + RI + \frac{1}{C} Q = E(t). \quad (32)$$

The units have been chosen so that 1 volt $= 1$ ohm $\cdot$ 1 ampere $= 1$ coulomb/1 farad $= 1$ henry $\cdot$ 1 ampere/1 second.

Substituting for $I$ from Eq. (31), we obtain the differential equation

$$LQ'' + RQ' + \frac{1}{C} Q = E(t) \quad (33)$$

for the charge $Q$. The initial conditions are

$$Q(t_0) = Q_0, \quad Q'(t_0) = I(t_0) = I_0. \quad (34)$$

Thus we must know the charge on the capacitor and the current in the circuit at some initial time $t_0$.

Alternatively, we can obtain a differential equation for the current $I$ by differentiating Eq. (33) with respect to $t$, and then substituting for $dQ/dt$ from Eq. (31). The result is

$$LI'' + RI' + \frac{1}{C} I = E'(t), \quad (35)$$

with the initial conditions

$$I(t_0) = I_0, \quad I'(t_0) = I'_0. \quad (36)$$

From Eq. (32) it follows that

$$I'_0 = \frac{E(t_0) - RI_0 - (I/C) Q_0}{L}. \quad (37)$$

Hence $I'_0$ is also determined by the initial charge and current, which are physically measurable quantities.

The most important conclusion from this discussion is that the flow of current in the circuit is described by an initial value problem of precisely the same form as the one that describes the motion of a spring–mass system. This is a good example of the unifying role of mathematics: Once you know how to solve second order linear equations with constant coefficients, you can interpret the results either in terms of mechanical vibrations, electric circuits, or any other physical situation that leads to the same problem.

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9 Gustav Kirchhoff (1824–1887), professor at Breslau, Heidelberg, and Berlin, was one of the leading physicists of the nineteenth century. He discovered the basic laws of electric circuits about 1845 while still a student at Königsberg. He is also famous for fundamental work in electromagnetic absorption and emission and was one of the founders of spectroscopy.