this explains why the series with parameter \( t \) are called “generalized” binomials and exponentials.

The following pairs of identities are valid for all real \( r \):

\[
\mathcal{B}_1(z)^r = \sum_{k \geq 0} \binom{tk + r}{k} \frac{r}{tk + r} z^k;
\]

\[
\mathcal{E}_1(z)^r = \sum_{k \geq 0} \frac{(tk + r)^{k-1}}{k!} z^k; \tag{5.60}
\]

\[
\frac{\mathcal{B}_1(z)^r}{1 - t + t \mathcal{B}_1(z)} = \sum_{k \geq 0} \binom{tk + r}{k} z^k;
\]

\[
\frac{\mathcal{E}_1(z)^r}{1 - tz \mathcal{E}_1(z)} = \sum_{k \geq 0} \frac{(tk + r)^k}{k!} z^k. \tag{5.61}
\]

(When \( tk + r = 0 \), we have to be a little careful about how the coefficient of \( z^k \) is interpreted; each coefficient is a polynomial in \( r \). For example, the constant term of \( \mathcal{E}_1(z)^r \) is \( r(0 + r)^{-1} \), and this is equal to 1 even when \( r = 0 \).

Since equations (5.60) and (5.61) hold for all \( r \), we get very general identities when we multiply together the series that correspond to different powers \( r \) and \( s \). For example,

\[
\mathcal{B}_1(z)^r \frac{\mathcal{B}_1(z)^s}{1 - t + t \mathcal{B}_1(z)} = \sum_{k \geq 0} \binom{tk + r}{k} \frac{r}{tk + r} z^k \sum_{i \geq 0} \binom{tk + s}{i} \frac{r}{tk + s} z^i = \sum_{n \geq 0} \sum_{k \geq 0} \binom{tk + r}{k} \frac{r}{tk + r} \binom{tk + s}{n - k} z^n.
\]

This power series must equal

\[
\frac{\mathcal{B}_1(z)^{r+s}}{1 - t + t \mathcal{B}_1(z)} = \sum_{n \geq 0} \binom{tn + r + s}{n} z^n,
\]

hence we can equate coefficients of \( z^n \) and get the identity

\[
\sum_{k} \binom{tk + r}{k} \binom{tn + r + s}{n - k} \frac{r}{tk + r} = \binom{tn + r + s}{n}, \quad \text{integer } n,
\]

valid for all real \( r \), \( s \), and \( t \). When \( t = 0 \) this identity reduces to Vandermonde’s convolution. (If by chance \( tk + r \) happens to equal zero in this formula, the denominator factor \( tk + r \) should be considered to cancel with the \( tk+r \) in the numerator of the binomial coefficient. Both sides of the identity are polynomials in \( r \), \( s \), and \( t \).) Similar identities hold when we multiply \( \mathcal{B}_1(z)^{r} \) by \( \mathcal{B}_1(z)^{s} \), etc.; Table 202 presents the results.