202 BINOMIAL COEFFICIENTS

Table 202 General convolution identities, valid for integer \( n \geq 0 \).

\[
\begin{align*}
\sum_k \binom{tk+r}{k} \binom{tn-tk+s}{n-k} \frac{r}{tk+r} &= \binom{tn+r+s}{n}, \\
(5.62) \\
\sum_k \binom{n}{k} (tk+r)^k (tn-tk+s)^{n-k} \frac{r}{tk+r} \frac{s}{tn-tk+s} &= \binom{tn+r+s}{n} \frac{r+s}{tn+r+s}, \\
(5.63) \\
\sum_k \binom{n}{k} (tk+r)^k (tn-tk+s)^{n-k} \frac{r}{tk+r} \frac{s}{tn-tk+s} &= (tn+r+s)^n. \\
(5.64) \\
\sum_k \binom{n}{k} (tk+r)^k (tn-tk+s)^{n-k} \frac{r}{tk+r} \frac{s}{tn-tk+s} &= (tn+r+s)^n. \\
(5.65)
\end{align*}
\]

We have learned that it’s generally a good idea to look at special cases of general results. What happens, for example, if we set \( t = 1 \)? The generalized binomial \( \mathcal{B}_1(z) \) is very simple—it’s just

\[
\mathcal{B}_1(z) = \sum_{k \geq 0} z^k = \frac{1}{1-z},
\]

therefore \( \mathcal{B}_1(z) \) doesn’t give us anything we didn’t already know from Vandermonde’s convolution. But \( \mathcal{E}_1(z) \) is an important function,

\[
\mathcal{E}_1(z) = \sum_{k \geq 0} (k+1)^k \frac{z^k}{k!} = 1 + z + \frac{3}{2}z^2 + \frac{8}{3}z^3 + \frac{125}{24}z^4 + \cdots
\]

(5.66)

that we haven’t seen before; it satisfies the basic identity

\[
\mathcal{E}_1(z) = e^{z\mathcal{E}[z]}
\]

(5.67)

This function, first studied by Eisenstein [75], arises in many applications.

The special cases \( t = 2 \) and \( t = -1 \) of the generalized binomial are of particular interest, because their coefficients occur again and again in problems that have a recursive structure. Therefore it’s useful to display these