series explicitly for future reference:

\[ B_2(z) = \sum_k \frac{(2k)}{k} \frac{z^k}{1+k} = \sum_k \frac{(2k+1)}{k} \frac{z^k}{1+2k} = \frac{1 - \sqrt{1-4z}}{2z}. \]  
\[ (5.68) \]

\[ B_{-1}(z) = \sum_k \frac{(1-k)}{k} \frac{z^k}{1-k} = \sum_k \frac{(2k-1)}{k} \frac{(-z)^k}{1-2k} = \frac{1 + \sqrt{1+4z}}{2z}. \]  
\[ (5.69) \]

\[ B_2(z)^r = \sum_k \frac{(2k+r)}{k} \frac{z^k}{1+2k} = \frac{\sqrt{1-4z}}{1+4z}. \]  
\[ (5.70) \]

\[ B_{-1}(z)^r = \sum_k \frac{(r-k)}{k} \frac{z^k}{1-r-k} = \frac{\sqrt{1+4z}}{1+4z}. \]  
\[ (5.71) \]

\[ \frac{B_2(z)}{\sqrt{1-4z}} = \sum_k \frac{(2k+r)}{k} z^k. \]  
\[ (5.72) \]

\[ \frac{B_{-1}(z)^{r+1}}{\sqrt{1+4z}} = \sum_k \frac{(r-k)}{k} z^k. \]  
\[ (5.73) \]

The coefficients \( \binom{1_n}{n+1} \) of \( B_2(z) \) are called the Catalan numbers \( C_n \), because Eugene Catalan wrote an influential paper about them in the 1830s [46]. The sequence begins as follows:

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_n )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>14</td>
<td>42</td>
<td>132</td>
<td>429</td>
<td>1430</td>
<td>4862</td>
<td>16796</td>
</tr>
</tbody>
</table>

The coefficients of \( B_{-1}(z) \) are essentially the same, but there’s an extra 1 at the beginning and the other numbers alternate in sign: \( 1, 1, -1, 2, -5, 14, \ldots \). Thus \( B_{-1}(z) = 1 + zB_2(-z) \). We also have \( B_1(z) = B_2(z)^1 \).

Let’s close this section by deriving an important consequence of (5.72) and (5.73), a relation that shows further connections between the functions \( B_{-1}(z) \) and \( B_2(-z) \):

\[ \frac{B_{-1}(z)^{n+1}}{\sqrt{1+4z}} = \sum_{k \leq n} \binom{n-k}{k} z^k. \]