It’s our old friend, the geometric series; \( F(\alpha', \ldots, \alpha_n; b, \ldots, b_n; z) \) is called hypergeometric because it includes the geometric series \( F(1,1;1;z) \) as a very special case.

The general case \( m = 1 \) and \( n = 0 \) is, in fact, easy to sum in closed form,

\[
F\left(\begin{array}{c}a_1 \\ 1\end{array}\right|z) = \sum_{k \geq 0} \binom{a_1}{k} \frac{z^k}{k!} = \sum_{k} \binom{a+k-1}{k} \frac{1}{(1-z)^a}, \quad (5.77)
\]

using (5.56). If we replace \( a \) by \(-a\) and \( z \) by \(-z\), we get the binomial theorem,

\[
F\left(\begin{array}{c}-a_1 \\ 1\end{array}\right|-z) = (1+z)^a
\]

A negative integer as upper parameter causes the infinite series to become finite, since \((-a)^k = 0\) whenever \( k > a \geq 0 \) and \( a \) is an integer.

The general case \( m = 0 \), \( n = 1 \) is another famous series, but it’s not as well known in the literature of discrete mathematics:

\[
F\left(\begin{array}{c}1 \\ b;1\end{array}\right|z) = \sum_{k \geq 0} \frac{(b-1)!}{(b-1+k)!} z^k \frac{1}{k!} \quad (5.78)
\]

This function \( \mathcal{I}_b \) is called a “modified Bessel function” of order \( b-1 \). The special case \( b = 1 \) gives us \( F(1,1;1|z) = I_0(2\sqrt{z}) \), which is the interesting series \( \sum_{k \geq 0} \frac{1}{k!} z^k \).

The special case \( m = n = 1 \) is called a “confluent hypergeometric series” and often denoted by the letter \( M \):

\[
F\left(\begin{array}{c}1 \\ a,b;1\end{array}\right) = \sum_{k \geq 0} \frac{\binom{a}{k} z^k}{k!} = M(a,b,z) \quad (5.79)
\]

This function, which has important applications to engineering, was introduced by Ernst Kummer.

By now a few of us are wondering why we haven’t discussed convergence of the infinite series (5.76). The answer is that we can ignore convergence if we are using \( z \) simply as a formal symbol. It is not difficult to verify that formal infinite sums of the form \( \sum_{k \geq n} \alpha_k z^k \) form a field, if the coefficients \( \alpha_k \) lie in a field. We can add, subtract, multiply, divide, differentiate, and do functional composition on such formal sums without worrying about convergence; any identities we derive will still be formally true. For example, the hypergeometric \( F\left(\begin{array}{c}1,1,1 \\ 1\end{array}\right|z) = \sum_{k \geq 0} \frac{k!}{k!} z^k \) doesn’t converge for any \textbf{nonzero} \( z \); yet we’ll see in Chapter 7 that we can still use it to solve problems. On the other hand, whenever we replace \( z \) by a particular numerical value, we do have to be sure that the infinite sum is well defined.