5.5 HYPERGEOMETRIC FUNCTIONS 207

The next step up in complication is actually the most famous hypergeometric of all. In fact, it was the hypergeometric series until about 1870, when everything was generalized to arbitrary m and n. This one has two upper parameters and one lower parameter:

\[
F\left(\begin{array}{c} a, b \\ c \end{array} \right | z) = \sum_{k=0}^{\infty} \frac{a^k b^k z^k}{c^k k!}.
\]  

(5.80)

It is often called the Gaussian hypergeometric, because many of its subtle properties were first proved by Gauss in his doctoral dissertation of 1812 [116], although Euler [95] and Pfaff [233] had already discovered some remarkable things about it. One of its important special cases is

\[
\ln(1+z) = z F\left(\begin{array}{c} 1, 1 \\ 2 \end{array} \right | -z) = \sum_{k=0}^{\infty} \frac{k! k!}{(k+1)! k!} \frac{(-z)^k}{k!}
\]

\[
= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots
\]

Notice that \(z^{-1}\ln(1+z)\) is a hypergeometric function, but \(\ln(1+z)\) itself cannot be hypergeometric, since a hypergeometric series always has the value 1 when \(z = 0\).

So far hypergeometrics haven’t actually done anything for us except provide an excuse for name-dropping. But we’ve seen that several very different functions can all be regarded as hypergeometric; this will be the main point of interest in what follows. We’ll see that a large class of sums can be written as hypergeometric series in a “canonical” way, hence we will have a good filing system for facts about binomial coefficients.

What series are hypergeometric? It’s easy to answer this question if we look at the ratio between consecutive terms:

\[
F\left(\begin{array}{c} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{array} \right | z) = \sum_{k=0}^{\infty} t_k, \quad t_k = \frac{a_1^k \ldots a_m^k z^k}{b_1^k \ldots b_n^k k!}
\]

The first term is \(t_0 = 1\), and the other terms have ratios given by

\[
\frac{t_{k+1}}{t_k} = \frac{a_1^{k+1} \ldots a_m^{k+1}}{b_1^{k+1} \ldots b_n^{k+1}} \frac{k! z^{k+1}}{(k+1)! z^k} \frac{z^{k+1}}{z^k} \frac{(k + a_1) \ldots (k + a_m) z}{(k + b_1) \ldots (k + b_n)(k+1)}.
\]

(5.81)

This is a rational function of \(k\), that is, a quotient of polynomials in \(k\). Any rational function of \(k\) can be factored over the complex numbers and put