into this form. The a’s are the negatives of the roots of the polynomial in the numerator, and the b’s are the negatives of the roots of the polynomial in the denominator. If the denominator doesn’t already contain the special factor \((k + 1)\), we can include \((k + 1)\) in both numerator and denominator. A constant factor remains, and we can call it \(z\). Therefore hypergeometric series are precisely those series whose first term is 1 and whose term ratio \(t_{k+1}/t_k\) is a rational function of \(k\).

Suppose, for example, that we’re given an infinite series with term ratio

\[
\frac{t_{k+1}}{t_k} = \frac{k^2 + 7k + 10}{4k^2 + 1},
\]

a rational function of \(k\). The numerator polynomial splits nicely into two factors, \((k + 2)(k + 5)\), and the denominator is \(4(k^2 + i/2)(k - i/2)\). Since the denominator is missing the required factor \((k + 1)\), we write the term ratio as

\[
\frac{t_{k+1}}{t_k} = \frac{(k + 2)(k + 5)(k + 1)(1/4)}{(k + i/2)(k - i/2)(k + 1)},
\]

and we can read off the results: The given series is

\[
\sum_{k \geq 0} t_k = t_0 F\left(\frac{2, 5, 1}{t/2, -i/2} \Big| 1/4\right).
\]

Thus, we have a general method for finding the hypergeometric representation of a given quantity \(S\), when such a representation is possible: First we write \(S\) as an infinite series whose first term is nonzero. We choose a notation so that the series is \(\sum_{k \geq 0} t_k\) with \(t_0 \neq 0\). Then we calculate \(t_{k+1}/t_k\). If the term ratio is not a rational function of \(k\), we’re out of luck. Otherwise we express it in the form (5.81); this gives parameters \(a_1, \ldots, a_n, b_1, \ldots, b_n\), and an argument \(z\), such that \(S = t_0 F(a_1, \ldots, a_n; b_1, \ldots, b_n; z)\).

Gauss’s hypergeometric series can be written in the recursively factored form

\[
F\left(\frac{a, b}{c} \Big| z\right) = 1 + \frac{a b}{1 c} z\left(1 + \frac{a+1 b+1}{2 c+1} z\left(1 + \frac{a+2 b+2}{3 c+2} z(1 + \cdots)\right)\right)
\]

if we wish to emphasize the importance of term ratios.

Let’s try now to reformulate the binomial coefficient identities derived earlier in this chapter, expressing them as hypergeometrics. For example, let’s figure out what the parallel summation law,

\[
\sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}, \text{ integer n},
\]

\(\text{(Now is a good time to do warmup exercise 11.)}\)