integer. We usually apply the parallel summation identity when \( r \) and \( n \) are positive integers; but then \(-n-r\) is a negative integer and the hypergeometric (5.76) is undefined. How then can we consider (5.82) to be legitimate? The answer is that we can take the limit of \( F(\frac{1}{1-r-n} \mid 1) \) as \( \epsilon \to 0 \).

We will look at such things more closely later in this chapter, but for now let’s just be aware that some denominators can be dynamite. It is interesting, however, that the very first sum we’ve tried to express hypergeometrically has turned out to be degenerate.

Another possibly sore point in our derivation of (5.82) is that we expanded \( \binom{n-k}{r-k} \) as \( (r+n-k)!/r!(n-k)! \). This expansion fails when \( r \) is a negative integer, because \((-m)!\) has to be \( \infty \) if the law

![Image](https://via.placeholder.com/150)

is going to hold. Again, we need to approach integer results by considering a limit of \( r + \epsilon \) as \( \epsilon \to 0 \).

But we defined the factorial representation \( \binom{r}{k} = r!/k!(r-k)! \) only when \( r \) is an integer! If we want to work effectively with hypergeometrics, we need a factorial function that is defined for all complex numbers. Fortunately there is such a function, and it can be defined in many ways. Here’s one of the most useful definitions of \( z! \), actually a definition of \( 1/z! \):

\[
\frac{1}{z!} = \lim_{n \to \infty} \left( \frac{z}{n} \right)^n \frac{z}{n}.
\]

(See exercise 21. Euler [81] discovered this when he was 22 years old.) The limit can be shown to exist for all complex \( z \), and it is zero only when \( z \) is a negative integer. Another significant definition is

\[
z! = \int_{0}^{\infty} t^z e^{-t} dt , \quad \text{if } \Re z > -1.
\]

This integral exists only when the real part of \( z \) exceeds -1, but we can use the formula

\[
z! = z(z-1)! \tag{5.85}
\]

to extend (5.84) to all complex \( z \) (except negative integers). Still another definition comes from Stirling’s interpolation of \( \ln z! \) in (5.47). All of these approaches lead to the same generalized factorial function.

There’s a very similar function called the Gamma function, which relates to ordinary factorials somewhat as rising powers relate to falling powers. Standard reference books often use factorials and Gamma functions simultaneously, and it’s convenient to convert between them if necessary using the

(We proved the identities originally for integer \( r \), and used the polynomial argument to show that they hold in general. Now we’re proving them first for irrational \( r \), and using a limiting argument to show that they hold for integers.)