19. Consider the vibrating system described by the initial value problem
\[ u'' + u = 3 \cos \omega t, \quad u(0) = 1, \quad u'(0) = 1. \]
(a) Find the solution for \( \omega \neq 1 \).
(b) Plot the solution \( u(t) \) versus \( t \) for \( \omega = 0.7, \omega = 0.8, \) and \( \omega = 0.9 \). Compare the results with those of Problem 18, that is, describe the effect of the nonzero initial conditions.

20. For the initial value problem in Problem 18 plot \( u' \) versus \( u \) for \( \omega = 0.7, \omega = 0.8, \) and \( \omega = 0.9 \); that is, draw the phase plot of the solution for these values of \( \omega \). Use a \( t \) interval that is long enough so the phase plot appears as a closed curve. Mark your curve with arrows to show the direction in which it is traversed as \( t \) increases.

Problems 21 through 23 deal with the initial value problem
\[ u'' + 0.125u' + u = F(t), \quad u(0) = 2, \quad u'(0) = 0. \]
In each of these problems:
(a) Plot the given forcing function \( F(t) \) versus \( t \) and also plot the solution \( u(t) \) versus \( t \) on the same set of axes. Use a \( t \) interval that is long enough so the initial transients are substantially eliminated. Observe the relation between the amplitude and phase of the forcing term and the amplitude and phase of the response. Note that \( \omega_0 = \sqrt{k/m} = 1 \).
(b) Draw the phase plot of the solution, that is, plot \( u' \) versus \( u \).

21. \( F(t) = 3 \cos(0.3t) \)
22. \( F(t) = 3 \cos t \)
23. \( F(t) = 3 \cos 3t \)
24. A spring–mass system with a hardening spring (Problem 32 of Section 3.8) is acted on by a periodic external force. In the absence of damping, suppose that the displacement of the mass satisfies the initial value problem
\[ u'' + u + \frac{1}{3}u^3 = \cos \omega t, \quad u(0) = 0, \quad u'(0) = 0. \]
(a) Let \( \omega = 1 \) and plot a computer-generated solution of the given problem. Does the system exhibit a beat?
(b) Plot the solution for several values of \( \omega \) between 1/2 and 2. Describe how the solution changes as \( \omega \) increases.

25. Suppose that the system of Problem 24 is modified to include a damping term and that the resulting initial value problem is
\[ u'' + \frac{1}{4}u' + u + \frac{1}{3}u^3 = \cos \omega t, \quad u(0) = 0, \quad u'(0) = 0. \]
(a) Plot a computer-generated solution of the given problem for several values of \( \omega \) between 1/2 and 2 and estimate the amplitude \( R \) of the steady response in each case.
(b) Using the data from part (a), plot the graph of \( R \) versus \( \omega \). For what frequency \( \omega \) is the amplitude greatest?
(c) Compare the results of parts (a) and (b) with the corresponding results for the linear spring.

REFERENCES

There are many books on mechanical vibrations and electric circuits. One that deals with both is: