Now \( \sin(x + y) = \sin x \cos y + \cos x \sin y \); so this ratio of sines is

\[
\frac{\cos 2n\pi \sin 2\epsilon \pi}{\cos n\pi \sin \epsilon \pi} = (-1)^n (2 + O(\epsilon)) ,
\]

by the methods of Chapter 9. Therefore, by (5.86), we have

\[
\lim_{\epsilon \to 0} \left(\frac{-n - \epsilon}{-2n - 2\epsilon}\right)! = 2(-1)^n \frac{\Gamma(2n)}{\Gamma(n)} = 2(-1)^n \frac{(2n - 1)!}{(n - 1)!} = (-1)^n \frac{(2n)!}{n!} ,
\]
as desired.

Let’s complete our survey by restating the other identities we’ve seen so far in this chapter, clothing them in hypergeometric garb. The triple-binomial sum in (5.29) can be written

\[
\begin{align*}
F\left( \begin{array}{c} 1 - a - 2n, 1 - b - 2n, 1 - 2n \\ a, b \end{array} \right) & = (-1)^n \frac{(2n)!}{n!} \frac{(a + b + 2n - 2)^n}{a^n b^n}, & \text{integer } n \geq 0.
\end{align*}
\]

When this one is generalized to complex numbers, it is called Dixon’s formula:

\[
F\left( \begin{array}{c} a, b, c \\ fc-a, 1 + c - b \end{array} \right) = \frac{(c/2)!}{c!} \frac{(c - a)x^{c/2}}{(c - a - b)^{c/2}}, \quad (5.96) \quad \Re a + \Re b < 1 + \Re c/2.
\]

One of the most general formulas we’ve encountered is the triple-binomial sum (5.28), which yields Saalschütz’s identity:

\[
\begin{align*}
F\left( \begin{array}{c} a, b, -n \\ c, a + b - c - n + 1 \end{array} \right) & = \frac{(c - a)^n (c - b)^n}{c^n (c - a - b)^n} \\
& = \frac{(a - c)^n (b - c)^n}{(-c)^n (a + b - c)^n}, & \text{integer } n \geq 0.
\end{align*}
\]

This formula gives the value at \( z = 1 \) of the general hypergeometric series with three upper parameters and two lower parameters, provided that one of the upper parameters is a nonpositive integer and that \( b_1 + b_2 = a_1 + a_2 + a_3 + 1 \). (If the sum of the lower parameters exceeds the sum of the upper parameters by 2 instead of by 1, the formula of exercise 25 can be used to express \( F(a_1, a_2, \ldots; b_1, b_2; 1) \) in terms of two hypergeometrics that satisfy Saalschütz’s identity.)

Our hard-won identity in Problem 8 of Section 5.2 reduces to

\[
\frac{1}{1 + x} F\left( \begin{array}{c} x + 1, n + 1, -n \\ 1, x + 2 \end{array} \right) = (-1)^n x^n x^{-n-1} .
\]