Sigh. This is just the special case $c = 1$ of Saalschiitz’s identity (5.97), so we could have saved a lot of work by going to hypergeometrics directly!

What about Problem 7? That extra-menacing sum gives us the formula

$$F\left(\frac{n+1}{2}, \frac{m-n}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}m+\frac{1}{2}, 1 \middle| 1 \right) = \frac{m}{n},$$

which is the first case we’ve seen with three lower parameters. So it looks new. But it really isn’t; the left-hand side can be replaced by

$$F\left(\frac{n}{2}, \frac{m-n-1}{2}, -\frac{1}{2} \middle| \frac{1}{2}m, \frac{1}{2}m-\frac{1}{2} \right)$$

using exercise 26, and Saalschiitz’s identity wins again.

Well, that’s another deflating experience, but it’s also another reason to appreciate the power of hypergeometric methods.

The convolution identities in Table 202 do not have hypergeometric equivalents, because their term ratios are rational functions of $k$ only when $t$ is an integer. Equations (5.64) and (5.65) aren’t hypergeometric even when $t = 1$. But we can take note of what (5.62) tells us when $t$ has small integer values:

$$F\left(\frac{1}{2}m, \frac{1}{2}m, -\frac{1}{2} \middle| \frac{1}{2}m, \frac{1}{2}m-\frac{1}{2} \right)$$

The first of these formulas gives the result of Problem 7 again, when the quantities $(r, s, n)$ are replaced respectively by $(1, 2n+1 - m, -1 - n)$.

Finally, the “unexpected” sum (5.20) gives us an unexpected hypergeometric identity that turns out to be quite instructive. Let’s look at it in slow motion. First we convert to an infinite sum,

$$\sum_{k\leq m} \binom{m+k}{k} \frac{1}{2^k} = 2$$

The term ratio from $(2m-k)!/m!(m-k)!$ is $2(k-m)/(k-2m)$, so we have a hypergeometric identity with $z = 2$:}

$$\binom{2m}{m} F\left(\frac{1}{2}, \frac{m}{2} \middle| \frac{1}{2}m, -\frac{1}{2}m \right) = 2^{2m}, \text{ integer } m \geq 0.$$ (5.98)