such that Eqs. (6) do not have a solution. Hence a necessary and sufficient condition for the existence of a solution of Eqs. (6) for arbitrary values of \( y_0, y'_0, \ldots, y_0^{(n-1)} \) is that the Wronskian

\[
W(y_1, \ldots, y_n) = \begin{vmatrix}
  y_1 & y_2 & \cdots & y_n \\
  y'_1 & y'_2 & \cdots & y'_n \\
  \vdots & \vdots & & \vdots \\
  y_0^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \\
\end{vmatrix}
\]

is not zero at \( t = t_0 \). Since \( t_0 \) can be any point in the interval \( I \), it is necessary and sufficient that \( W(y_1, y_2, \ldots, y_n) \) be nonzero at every point in the interval. Just as for the second order linear equation, it can be shown that if \( y_1, y_2, \ldots, y_n \) are solutions of Eq. (4), then \( W(y_1, y_2, \ldots, y_n) \) is either zero for every \( t \) in the interval or else is never zero there; see Problem 20. Hence we have the following theorem.

**Theorem 4.1.2**

If the functions \( p_1, p_2, \ldots, p_n \) are continuous on the open interval \( I \), if the functions \( y_1, y_2, \ldots, y_n \) are solutions of Eq. (4), and if \( W(y_1, y_2, \ldots, y_n)(t) \neq 0 \) for at least one point in \( I \), then every solution of Eq. (4) can be expressed as a linear combination of the solutions \( y_1, y_2, \ldots, y_n \).

A set of solutions \( y_1, \ldots, y_n \) of Eq. (4) whose Wronskian is nonzero is referred to as a fundamental set of solutions. The existence of a fundamental set of solutions can be demonstrated in precisely the same way as for the second order linear equation (see Theorem 3.2.5). Since all solutions of Eq. (4) are of the form (5), we use the term general solution to refer to an arbitrary linear combination of any fundamental set of solutions of Eq. (4).

The discussion of linear dependence and independence given in Section 3.3 can also be generalized. The functions \( f_1, f_2, \ldots, f_n \) are said to be linearly dependent on \( I \) if there exists a set of constants \( k_1, k_2, \ldots, k_n \), not all zero, such that

\[
k_1 f_1 + k_2 f_2 + \cdots + k_n f_n = 0
\]

for all \( t \) in \( I \). The functions \( f_1, \ldots, f_n \) are said to be linearly independent on \( I \) if they are not linearly dependent there. If \( y_1, \ldots, y_n \) are solutions of Eq. (4), then it can be shown that a necessary and sufficient condition for them to be linearly independent is that \( W(y_1, \ldots, y_n)(t_0) \neq 0 \) for some \( t_0 \) in \( I \) (see Problem 25). Hence a fundamental set of solutions of Eq. (4) is linearly independent, and a linearly independent set of \( n \) solutions of Eq. (4) forms a fundamental set of solutions.

**The Nonhomogeneous Equation.** Now consider the nonhomogeneous equation (2).

\[
L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = g(t).
\]

If \( Y_1 \) and \( Y_2 \) are any two solutions of Eq. (2), then it follows immediately from the linearity of the operator \( L \) that

\[
L[Y_1 - Y_2](t) = L[Y_1](t) - L[Y_2](t) = g(t) - g(t) = 0.
\]

Hence the difference of any two solutions of the nonhomogeneous equation (2) is a solution of the homogeneous equation (4) Since any solution of the homogeneous