where \(y_1\) and \(y_2\) are \(n\) times differentiable functions and \(c_1\) and \(c_2\) are arbitrary constants. Hence, show that if \(y_1, y_2, \ldots, y_n\) are solutions of \(L[y] = 0\), then the linear combination \(c_1y_1 + \cdots + c_ny_n\) is also a solution of \(L[y] = 0\).

19. Let the linear differential operator \(L\) be defined by

\[ L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y, \]

where \(a_0, a_1, \ldots, a_n\) are real constants.
(a) Find \(L[e^t]\).
(b) Find \(L[e^{rt}]\).
(c) Determine four solutions of the equation \(y^{iv} - 5y'' + 4y = 0\). Do you think the four solutions form a fundamental set of solutions? Why?

20. In this problem we show how to generalize Theorem 3.3.2 (Abel’s theorem) to higher order equations. We first outline the procedure for the third order equation

\[ y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0. \]

Let \(y_1, y_2,\) and \(y_3\) be solutions of this equation on an interval \(I\).
(a) If \(W = W(y_1, y_2, y_3)\), show that

\[ W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}. \]

\(W'\) is the derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.
(b) Substitute for \(y_1'', y_2'',\) and \(y_3''\) from the differential equation; multiply the first row by \(p_1\), the second row by \(p_2\), and add these to the last row to obtain

\[ W' = -p_1(t)W. \]

(c) Show that

\[ W(y_1, y_2, y_3)(t) = c \exp \left[ -\int p_1(t) \, dt \right]. \]

It follows that \(W\) is either always zero or nowhere zero on \(I\).
(d) Generalize this argument to the \(n\)th order equation

\[ y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \]

with solutions \(y_1, \ldots, y_n\). That is, establish Abel’s formula,

\[ W(y_1, \ldots, y_n)(t) = c \exp \left[ -\int p_1(t) \, dt \right], \]

for this case.

In each of Problems 21 through 24 use Abel’s formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

21. \(y'''' + 2y'' - y' - 3y = 0\)
22. \(y'' + y = 0\)
23. \(ty''' + 2y'' - y' + ty = 0\)
24. \(t^2 y'' + ty''' + y'' - 4y = 0\)

25. The purpose of this problem is to show that if \(W(y_1, \ldots, y_n)(t_0) \neq 0\) for some \(t_0\) in an interval \(I\), then \(y_1, \ldots, y_n\) are linearly independent on \(I\), and if they are linearly independent and solutions of

\[ L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0 \]

on \(I\), then \(W(y_1, \ldots, y_n)\) is nowhere zero in \(I\).