As Example 1 illustrates, the procedure for solving an \( n \)th order linear differential equation with constant coefficients depends on finding the roots of a corresponding \( n \)th degree polynomial equation. If initial conditions are prescribed, then a system of \( n \) linear algebraic equations must be solved to determine the proper values of the constants \( c_1, \ldots, c_n \). While each of these tasks becomes much more complicated as \( n \) increases, they can often be handled without difficulty with a calculator or computer.

For third and fourth degree polynomials there are formulas, analogous to the formula for quadratic equations but more complicated, that give exact expressions for the roots. Root-finding algorithms are readily available on calculators and computers. Sometimes they are included in the differential equation solver, so that the process of factoring the characteristic polynomial is hidden and the solution of the differential equation is produced automatically.

If you are faced with the need to factor the characteristic polynomial by hand, here is one result that is sometimes helpful. Suppose that the polynomial

\[
a_0 r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0
\]

has integer coefficients. If \( r = p/q \) is a rational root, where \( p \) and \( q \) have no common factors, then \( p \) must be a factor of \( a_n \) and \( q \) must be a factor of \( a_0 \). For example, in Eq. (8) the factors of \( a_0 \) are \( \pm 1 \) and the factors of \( a_n \) are \( \pm 1, \pm 2, \pm 3, \) and \( \pm 6 \). Thus, the only possible rational roots of this equation are \( \pm 1, \pm 2, \pm 3, \) and \( \pm 6 \). By testing these possible roots, we find that \( 1, -1, 2, \) and \( -3 \) are actual roots. In this case there are no other roots, since the polynomial is of fourth degree. If some of the roots are irrational or complex, as is usually the case, then this process will not find them, but at least the degree of the polynomial can be reduced by dividing out the factors corresponding to the rational roots.

If the roots of the characteristic equation are real and different, we have seen that the general solution (5) is simply a sum of exponential functions. For large values of \( t \) the

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3The method for solving the cubic equation was apparently discovered by Scipione dal Ferro (1465–1526) about 1500, although it was first published in 1545 by Girolamo Cardano (1501–1576) in his *Ars Magna*. This book also contains a method for solving quartic equations that Cardano attributes to his pupil Ludovico Ferrari (1522–1565). The question of whether analogous formulas exist for the roots of higher degree equations remained open for more than two centuries, until in 1826 Niels Abel showed that no general solution formulas can exist for polynomial equations of degree five or higher. A more general theory was developed by Evariste Galois (1811–1832) in 1831, but unfortunately it did not become widely known for several decades.